

ART

THE KEY TO UNDERSTANDING THE STRUCTURE OF THOUGHT

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If I am to know an object, though I need not  
know its external properties, I must know all  
its internal properties  
Ludwig Wittgenstein

## INTRODUCTION

The relationship between music and mathematics has given rise to many proposals. Some books develop interesting and complex ideas that are more focused on acoustic physics and not on the composing process itself. Other authors propose certain results that seem to be very inaccurate, and out of context.

This essay highlights a method of musical interpretation, in which music can be understood as a vector. A similar model was designed by the famous composer Iannis Xenakis in several of his compositions. The difference is that Xenakis' methods create musical results from mathematics, and mathematics is not used as a tool to explain the musical phenomenon. The idea of a vector is taken from Xenakis, but extended to provide a more representative way of analyzing musical behavior.

This essay is not about topics of acoustic physics, as most texts on the market, but exposes the way in which the musician manipulates musical ideas. The environmental properties of the place where music is interpreted or the distinctive timbre of each instrument is not covered by this essay.

Another aspect relating to music is the subjectivity of musical understanding, or how subjective it is. These are matters that relate to aesthetics and are not covered here.

What matters is to explain how the different variables interact with each other and the patterns that can be found in it. Due to lack of certain mathematical programs some equations can not be expressed in graphs, e.g. the parametric ones.

The meaning of this will arise from establishing the qualities of the note and seeing how it can be analyzed. The vector variables are moved within a space that has certain characteristics. A three-dimensional description of the note is able to approximate to any model, as a result of this, a wide range of options can be used from this idea. You can even expand vector manipulation and use some of its properties to design more complex musical structures.

There has not been an effective break with tradition, there are still ties. Even today musicians are subject to traditional ideas. The curve in music has a similarity to curve forms in painting. This aspect is noted at the end of the text as an additional proposal: seen from a certain perspective there are points in which all arts converge. The last chapter discusses parallel themes that seem to have a relationship to the vector concept in music.

## THE NOTE AND ITS VARIABLES

This section begins with an excerpt from the doctoral thesis of (Dra. Susana Tiburcio. Page 83, section 4.2).

### 4.2 Musical Parameters

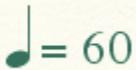
Sound entities can be represented as a vector of four independent variables:  $E = (c, h, g, u)$  where  $c$  is the timbre or instrumental family,  $h$  is the pitch of sound,  $g$  is the intensity of sound or dynamic form and  $u$  is the duration of the sound. As an example we take the following situation:

Let  $E$  be the vector whose components:  $c$  - violin,  $h$  is a  $C$  in  $C3$ ,  $g4$  is a forte and  $u$  is a  $\frac{1}{4}$  sec.

$E = (\text{violin}, C3, \text{forte}, 1/4\text{sec})$ . We can see a musical composition as a sequence of vectors  $E$ , which is denoted by the pair  $(t, E)$  following the lexicographical order in a space of five dimensions [Xenakis, Xenakis].

On the information above we can obtain the following conclusions:

- 1) Sound entities are represented by  $E = (c, h, g, u)$  or a vector of four variables.
- 2)  $c, h, g,$  and  $u$  are variables. So  $c$  can be a violin,  $h$  can be a  $C$ , in this case  $C3$ ,  $g4$  is a  $f$  (forte) and  $u$  is a  $\frac{1}{4}$  s (quarter of a second) or  $0.25$ s. This last duration could be in the case that we have:



where the duration expressed would be equivalent to a sixteenth note. This is an issue that we will analyze later, but we just mention it here in passing. This model establishes the duration of the note, but does not establish where it is located in space. There is an ambiguity.

From the previous paragraph it is concluded that the method of Xenakis works to develop music but does not allow generalizing to other older musical techniques. Thus each entry in the vector  $E$ , either  $c, h, g$  or  $u$  could include a wide range of inputs. This arrangement can also change, so  $c$  could be a flute,  $h$   $A$  ( $A5$ ) ( $880\text{hz}$ )  $g$  could be a  $mp$  (mezzopiano) and  $u$  could be a triplet ( $1/3$ ).

The above analysis leads to a series of questions and possible answers that are listed below.

- 1) What is the basic unit of music?

We can say that the basic unit is the note as a letter will be to written language. The note has a frequency, sound intensity or dynamic and a rhythm that is defined as the location and duration. We speak of a note as it is, nothing more. All differences produced by each instrument due to its construction and materials would be the timbre, and, in this, we must take into account the harmonic series accompanying the note.

Also in the musical construction harmony, counterpoint and rhythm are independent of the timbre. So a composer may think that a musical passage will be played by a certain instrument,

but it could be played by other one, and harmony and counterpoint will be comprehensible as well.

Composers like Korsakov, in his orchestration book, recommend doubling certain notes of the chord, or combinations of instruments which will produce a particular effect. It can be cheerful, dull, sharp and he recommended certain duplications. In reality this serves for a certain type of music, or as a reference for the student of composition. A composer can duplicate any instruments while the resulting sound is representative of what he seeks to achieve. So a combination of instruments that is not in the Korsakov's orchestration treatise, or other orchestration book, is not necessarily uninteresting or lacking in validity, it is just a combination that was not considered by him. The timbre selection is a matter of taste and is not part of the treatment we give to the note, which is our main concern; we will deal only pure frequencies. The place in which someone could play a note or sound will affect its result. Playing the same note in a room of 9 square meters closed, or into a large auditorium create different effects, as projection of sound changes.

2) Xenakis, in this model has a four-dimensional vector. The new vector that will be used, will be a three-dimensional one, which certainly differs from the variables mentioned in Xenakis model. Its very important to mention that, Xenakis in his work called *Metastaseis*, he presents pitch on the Y axis and time on the X axis or the UPIC system uses the same two-dimensional system for creating music from drawings. A model similar to this one with a additional axis and some differences, is able to represent the music at any period or style, as all the musical notes are a sign that represents a pitch, dynamic and rhythm.

A four-dimensional model, could consider the timbre or articulations, but we can not think about this as an additional variable, because articulations as stated, are combinations of the other elements. Timbre gives its particular sound because of the construction and materials used on the instrument. Also the place where an instrument is played affects how the player will have to account for his environment. This are matters of Acoustic Physics.

For practical purposes of this essay, certain examples have dynamics omitted, as in many cases they are constant over long sections and it is simpler to explain certain processes using two-dimension equations. Processes are also simpler to express using natural numbers associated with Y axis, using the frequencies, but this would affect the result. They make this look like linear behavior but music has a lack of linear structures.

3) Duration is a property of time, and what Xenakis uses in his model responds only to the duration of the note, but it does not indicate where it starts or where is located in space. Actually, a note duration will be the space between that note and the one that that follows it. And if we want to know that duration in seconds, it is as simple as to apply a formula that will be seen in another chapter.

The answer to the above questions and the development of those ideas will be the basis of this essay. Distribution of vector entries were chosen to be capable of generating all kinds of sound, in terms of their location in space, frequency and dynamics. This model not only allows to

generalize the method represented by equations and graphs, as also is able to approximate any kind of acoustic music from Palestrina to Schoenberg and beyond.

In the conclusions section we will briefly expand on other artistic disciplines to which this method can relate.

From this point we will introduce the names of all variables that will be used in the vector. To be able to establish it, first we need to define point by point each of the variables.

We have:

f = frequency or pitch: This tells the note in question, so A4 will be 440Hz. This will be located on the y-axis.

I = sound intensity or nuance: for example p, mp, mf, f etc.. Although this is an undefined variable as it varies between performers and in turn depends on the conditions of the place where we can use approximate values or assumptions, but we could also apply the principles laid down for twelve-tone music and say, pppp = 1 ppp = 2, p = 3, p = 4, mp = 5 and so on. These will be located on z-axis.

r = the rate (contextual duration): is the location of the note in time. It is located on the x axis and can represent any given value. The note length is obtained from the value which defines the amount of bpm (beats per minute) and the type of figure that is. This will be located on x-axis.

**On this essay the number following the note refers to the octave which is placed, so for example if it says A7 means A on the 7<sup>th</sup> octave, not that it is a chord with a seventh interval.**

# TONE

Each note has an associated mechanical wave, with certain characteristic properties, such as amplitude, frequency, wavelength, cycle and others that belong to the study of acoustic physics. In musical terms the wave property most often used to refer to tone is its frequency and this is the unit we use from now on. It is usual, as we will see in the table below, to use names exclusively applied to 12 notes, going from C to B and then repeating in a  $2^n$  pattern. C1 in the first octave has a value of 32.7. Applying formula  $32.7 \times 2^n$ , then for  $n=(0,1,2,3,4,5,6,7,8,9)$  will obtain each octave C. Evaluating that formula in  $n=1$   $32.7 \times 2^1$ . This result will be C2 (65.41Hz).

The frequencies used are not the only ones that exist. In microtonal music created in the twentieth century is a very interesting type of music, but limited to the capabilities of the performer. It is not so “virtuoso”. In the case of stringed instruments for example, the smaller the range, for example quarter tone, eighth tone, then it is more likely to fail at high speeds which makes hearing the microtones less interesting. Only a few composers like Julián Carrillo, Pierre Boulez, John Eaton, Sofia Gubaidulina, and a few others were devoted largely to this aspect of music and even designed microtonal instruments, like Carrillo.

Frequency (in hertz) of musical notes									
	Oc. 0	Oc. 1	Oc. 2	Oc. 3	Oc. 4	Oc. 5	Oc. 6	Oc. 7	Oc. 8
<b>C</b>		32,7	65,41	130,8	261,6	523,3	1047	2093	4186
<b>C#</b>		34,65	69,3	138,6	277,2	554,4	1109	2217,5	
<b>D</b>		36,71	73,42	146,8	293,7	587,3	1175	2349,3	
<b>D#</b>		38,89	77,78	155,6	311,1	622,3	1245	2489	
<b>E</b>		41,2	82,41	164,8	329,6	659,3	1319	2637	
<b>F</b>		43,65	87,31	174,6	349,2	698,5	1397	2793,8	
<b>F#</b>		46,25	92,5	185	370	740	1480	2960	
<b>G</b>		49	98	196	392	784	1568	3136	
<b>G#</b>		51,91	103,83	207,7	415,3	830,6	1661	3322,4	
<b>A</b>	27,5	55	110	220	440	880	1760	3520	
<b>La#</b>	29,14	58,27	116,54	233,1	466,2	932,3	1865	3729,3	
<b>B</b>	30,87	61,74	123,47	246,9	493,9	987,8	1976	3951,1	

Now that a note and its octave  $2^n$  keep a relationship does not mean they are exactly the same note, just keep a relation of  $2^n$  as already mentioned.

The frequencies in music are linked in two ways:

a) The melody: this could be described as notes moving horizontally. This aspect has been studied a great deal in counterpoint technique, which analyzes the relationship of music to different voices and how they interact.

b) Harmony: is the vertical relationship of the notes that form chords. The same chord has several inversions.

A property of all sets is their ability to permute. And as the tonalities, or dodecaphonic series are subsets of the main set, then you can call the melody the number of possible horizontal permutations and harmony the vertical number of permutations possible.

## **Rhythm (contextual duration)**

The duration of a note in terms of seconds is related to the beats per minute that the composer may establish. For example “andante” and “presto” will produce durations in seconds of the quarter note, where one could be the double of the other. A quarter note played in “presto” (180-200ppm) might be an eighth note played in “Andante” (76-108ppm), taking the unit of measurement that designates tempo as the quarter note. The time signature determines which note value is used as the unit of bpm (beats per minute). There are cases where the metrics are constantly changing and it must be clarified which note will follow as the time unit or bpm. One of the questions that are raised about the length of the note may be solved with a simple

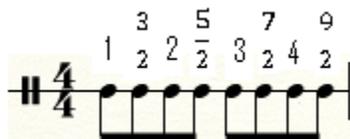
equation,  $d = \frac{60s}{b}$  / with d = length b = beats per minute. Then we can say that in a piece played at 60 bpm (b=60 bpm) we have  $d = \frac{60s}{60ppm} = 1$  that will be the quarter note (note value) that sets the pulse which lasts for one second and then applying simple fractions to 60 bpm, then eighth lasts 0.5s, the sixteenth 0.25s the thirty-second note is 0.125s.

If we take the quarter note as the unit that measures beats, for which the beats per minute apply, taking the example of a bar in 4/4 then we will obtain these results. If the time signature specifies that a half note is the unit of time then we take it as the representative unit of bpm. So if we say that a half note is equal to 60ppm then the half note will be 1 second, the quarter note 0.5s, the eighth 0.25s and the sixteenth 0.125 s. In other words the speed is increased in a ratio again of 2n. Importantly, both the tempo and frequency variables tend to be more accurate while dynamics are more relative.

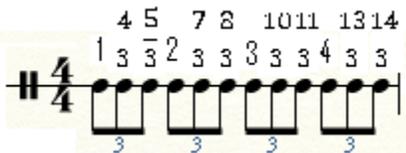
Note	Duration	Symbol
whole note	four beats	
	4	
half note	two beats	
	2	
quarter note	one beat	
	1	
eighth note	half of a beat	
1/2		
sixteenth note	quarter of a beat	
1/4		
thirty-second note	sixteenth of a beat	
1/8		

We can say that this is the duration of each note, in musical terms. In practical terms we have to know the bpm to apply the formula and obtain results in terms of seconds. Now the issue is how to represent each note numerically. For example:

A quarter note lasts one beat and an eighth note lasts half a beat. Under this condition and as a measure of 4/4 has eight eighth notes, numbers associated to them are shown in the next figure:



Another example is the case of triplets also in a 4/4 bar:



In both example using a bar in 4/4 each note occupies a position in the coordinate axis T, and this can be represented numerically. Both examples are interesting but we have not yet mentioned silence since rests are a fundamental aspect of music. Rests are segments in which performer does not play, and are part of the musical articulation, even speech. When we speak we use a breathing space between sentences, it is a necessity in language. Maybe this need to rest so you can recover breath is transmitted to music. We have to stop so we can breath and continue talking or let someone else talk while we take a breath. In music it is not always a necessity,

since many instruments can play for hours and the performer does not need to make any pause. Now, rests and how they are represented will be shown in the next bar examples and T axis representations:

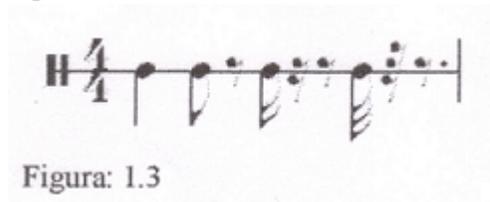


Figura: 1.3

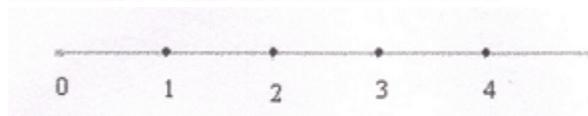
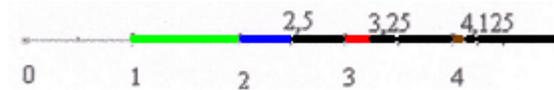


Figura: 1.2

Let us use the quarter note as the reference beat, so all notes from the previous example will match with it.

If we represent the notes including their duration we can see the notes and how long they last in the colored sections. Applying the aforementioned formula to obtain the length assuming that the quarter note is equivalent to 60 pulses per second they can be represented as follows:



The lines in black are rests and their duration, the eighth, the sixteenth and thirty-second sounding notes are in green, blue, red and brown. Black and silence in visual arts and music are in some way synonymous, because one is the total lack of sound and the other is the lack of light reflection.

The (bpm) [bits per minute] is the speed of executing the unit of time established that composers use as the main reference for tempo, it is defined at the beginning of the bar (measure). Another feature of the bar is its meter, which defines the amount of notes and which one will be used as the bpm reference unit. A 4/4 meter uses the quarter note as the reference to set bpm 8/8 would be 8 eighths, and eighth note will be the reference one, or could be a dotted note, quarter note with a dot. The meter will be a fraction, its numerator means the number of notes and the denominator will be the type of note value used as reference.

1	Whole note
2	Half note
4	Quarter note
8	Eight note
16	Sixteenth note
32	Thirty-second note

The above table contains the numbers used as denominators, associated with each type of note value. Composers will use one related to their needs, so they can develop the rhythms that match his or her ideas. Using the formula we can say how long a note value will last in terms of seconds.

Within the time signature, there are rules for accents, taking a very classical perspective, yet the composer can choose whether certain notes are emphasized, or if he establishes additional references in his work, explaining that the time signature will use certain accents that perpetuate throughout the work.

## **Expression**

The third and last of the variables to explain and use in this model will be the expression (intensity) that will refer to the loudness of the note that will be played. From the physical point of view, the sound intensity is measured in decibels. As already mentioned, of the three variables, this could be the least accurate and is very dependent upon the place and the acoustics of the place music is played. So the interpreter must adapt himself to each environment and its particular acoustic characteristics. There are differences between performing a solo or playing with an orchestra.

The absolute threshold of hearing represents the minimum amount of sound or vibrations per second required for sound for the human ear to be able to perceive. That number corresponds to a vibration frequency of about 1 kHz (10-12 W/m<sup>2</sup>).

Unlike the tone (pitch) that is more accurate, dynamics tend to be less accurate and there is not a universal scale for decibel levels related to dynamics such as p, mp, mf etc. Instead acoustic differences vary. A musical motif that should be played in *mp* has not an exact value in decibels. This could be an interesting study for a physicist or a sound engineer specialized in acoustics. They can analyze how a performer plays dynamics in different environments. The most used and common symbols are represented in the table below, there are more but this will give an idea.

<i>fff</i>	LOUDEST POSSIBLE
<i>ff</i>	VERY LOUD
<i>f</i>	LOUD
<i>mf</i>	MODERATELY LOUD
<i>mp</i>	MODERATELY SOFT
<i>p</i>	SOFT
<i>pp</i>	VERY SOFT
<i>ppp</i>	SOFTEST POSSIBLE

Dynamics do not have a defined parameter and they are relative.

A list of some decibel sources:

Source	Intensity	Intensity Level	# of Times Greater Than TOH
Threshold of Hearing (TOH)	$1 \cdot 10^{-12} \text{ W/m}^2$	0 dB	$10^0$
Rustling Leaves	$1 \cdot 10^{-11} \text{ W/m}^2$	10 dB	$10^1$
Whisper	$1 \cdot 10^{-10} \text{ W/m}^2$	20 dB	$10^2$
Normal Conversation	$1 \cdot 10^{-6} \text{ W/m}^2$	60 dB	$10^6$
Busy Street Traffic	$1 \cdot 10^{-5} \text{ W/m}^2$	70 dB	$10^7$
Vacuum Cleaner	$1 \cdot 10^{-4} \text{ W/m}^2$	80 dB	$10^8$
Large Orchestra	$6.3 \cdot 10^{-3} \text{ W/m}^2$	98 dB	$10^{9.8}$
Walkman at Maximum Level	$1 \cdot 10^{-2} \text{ W/m}^2$	100 dB	$10^{10}$
Front Rows of Rock Concert	$1 \cdot 10^{-1} \text{ W/m}^2$	110 dB	$10^{11}$
Threshold of Pain	$1 \cdot 10^1 \text{ W/m}^2$	130 dB	$10^{13}$
Military Jet Takeoff	$1 \cdot 10^2 \text{ W/m}^2$	140 dB	$10^{14}$
Instant Perforation of Eardrum	$1 \cdot 10^4 \text{ W/m}^2$	160 dB	$10^{16}$

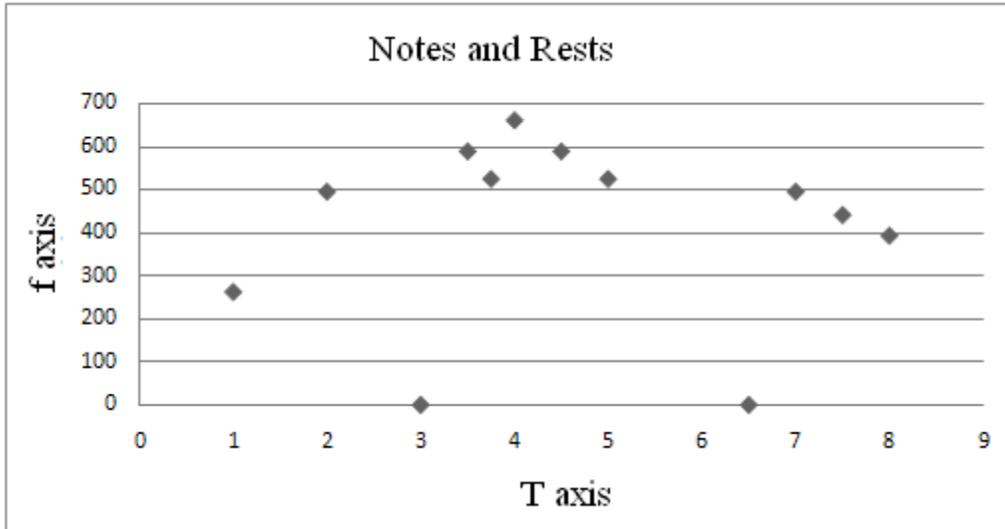
A higher level of accuracy for decibels would require computer manipulation.

## SILENCE

The musical silence (rest) implies an absence of frequency and loudness. Although it occupies a position in time. Is represented by a value in T, as f and I are both set at zero.

The rest is necessary in the case of brass and woodwind, so they can catch their breath and continue to perform. Others do not need it but the music from time to time requires it.



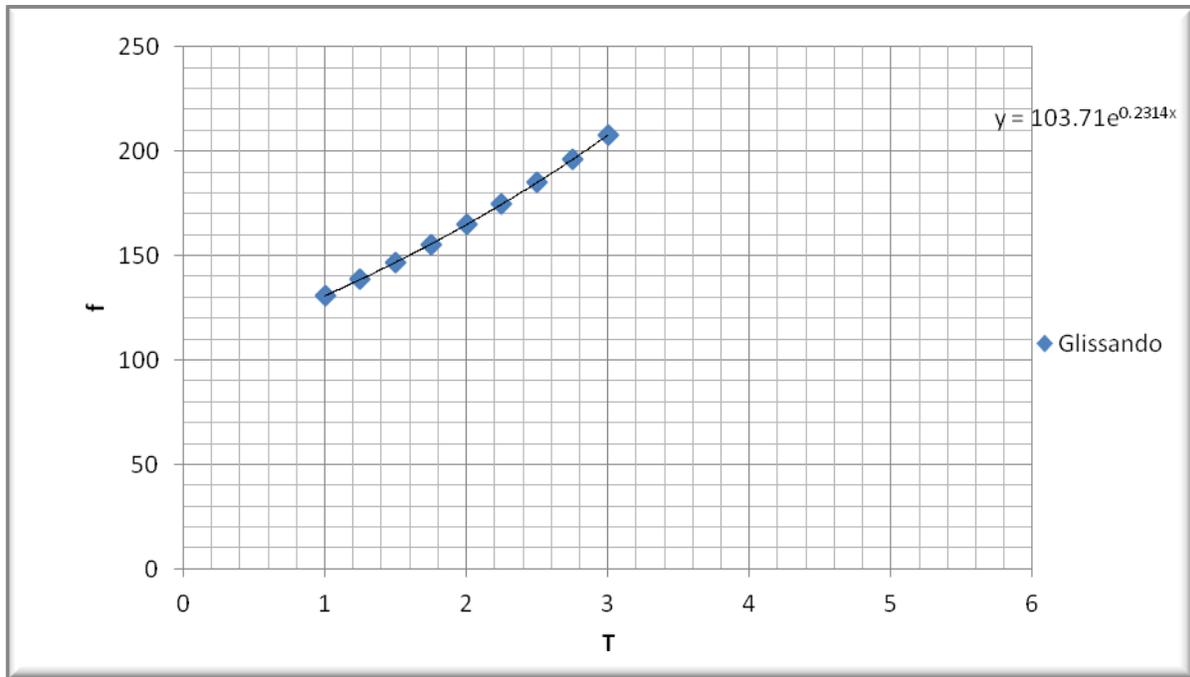


## GLISSANDO

So far, notes have been treated as points in a three-dimensional space, however there is a particular case in which discrete values can not represent a specific behavior, the glissando. It is a continuous frequency range. This case affects the frequency variable and the displacement (or rhythm) of the note. Both variables are continuous as the sound intensity remains the same. We speak of a closed interval comprising all intermediate values, which gives a continuous curve. The interval  $[a, b]$  meaning all the values between point a and point b are included in what the musician performs. In this case of a A4 (440hz) a E5 (659.3Hz).



Represented on the graph as follows:



A music scale it is a curve, in music only a few points of that curve are used. The big difference with the glissando is that the performer slides his finger from one note to another, playing all the notes between them.

## ARTICULATIONS

Articulations are part of the musical language and although some people take them as an independent variable, they are not. They represent a simplification of musical language, blends of the aforementioned variables. Articulations allow the composer to provide the performer with a better understanding of the score. Ornaments can be analyzed in terms of the other 3 variables. It is important to understand that using a staccato is not the same as using a note followed by lots of rests. Using lots of rests causes saturation which is unnecessary and difficult for the performer to read. Using a tie is not the same as designing new shapes to represent the note durations that we need. We can also consider the trill: if you write it out note for note it will be quite dense and cumbersome.

In order for the musical language to be comprehensible some simplification is required. An excessive saturation of note values or dynamics tends to be annoying for the performer and is a problem of some modern composers who design their own language sometimes with a lack of accuracy. This lends itself to misunderstanding of the terminology used. It is not a priority of this work to do a comprehensive analysis of all the symbols used as articulations. Someone else may like to do the task.

## Behaviors

We can say that an equation that describes all three variables would be similar to a position vector in physics.

This would be of the form  $N = T(p) i + f(p) j + I(p) k$  where:

$N$  = the name its assign to the vector we are referring to.

$T(p)$  = position in time

$f(p)$  = frequency

$I(p)$  = Intensity

Each of the variables on the right side of the equation can be replaced by a formula in terms of  $p$  that is capable of describing the desired displacement of the note in that variable. The resulting equation would be a curve in a three-dimensional space. The idea is to evaluate a set of values for  $(p)$  within the set of natural numbers, and in this way obtain the path of the note through a sequence of numbers that limit the solution set of the equation. Returning to the case of articulations we will represent a trill going from C4 to D4.

Their values were:

n	T(n)	f(n)	I(n)
0	1	261.6	30
1	1.125	293.7	30
2	1.25	261.6	30
3	1.375	293.7	30
4	1.5	261.6	30
5	1.625	293.7	30
6	1.75	261.6	30
7	1.875	293.7	30

The equation that represents numbers in the above table is of the form:

$N = \left(\frac{n}{8} + 1\right)i + \left(32.1\sin\left(\frac{n\pi}{2}\right) + 261.6\right)j + 30k$  where  $n = 0$  produces the first note,  $n = 1$  the second and so on.

In musical language this would be as follows:



Sometimes in music, a motif, as we call a segment of musical syntax, can move to another tonal center. This seems simple but the music is not linear then if we take two ascending scales and put them over one another, they are seen to be an exponential curve.

The structure of the major scale construction follows a pattern that is:

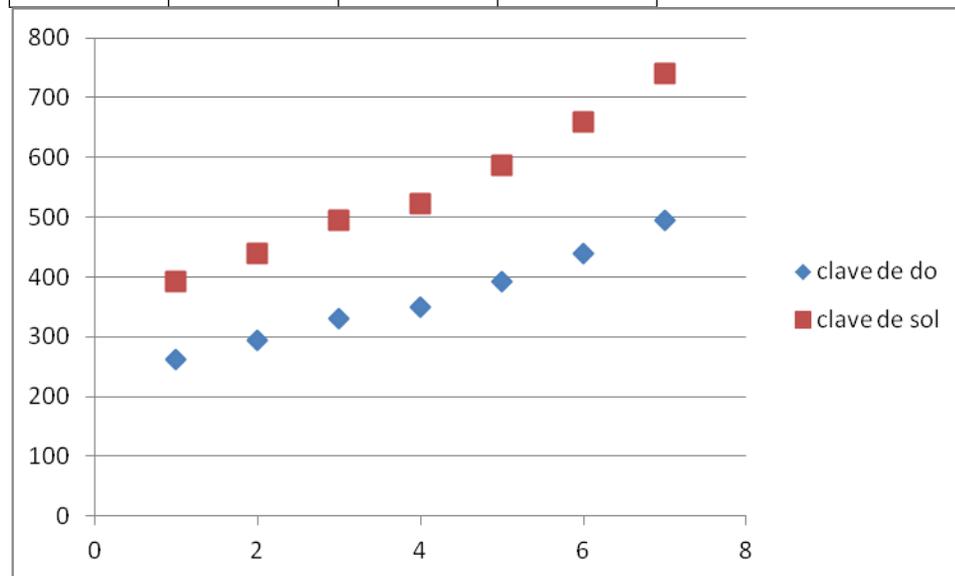
tone	tone	semitone	tone	tone	tone	semitone
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Ways to arrange tones are many, pentatonic scales, whole tone scales, major scales and others.

Next examples are in Ionian Mode (major) t-t-s-t-t-t-s



C major scale		G major scale	
T	f	T	f
1	261.6	1	392
2	293.7	2	440
3	329.6	3	493.9
4	349.2	4	523.3
5	392	5	587.3
6	440	6	659.3
7	493.9	7	740



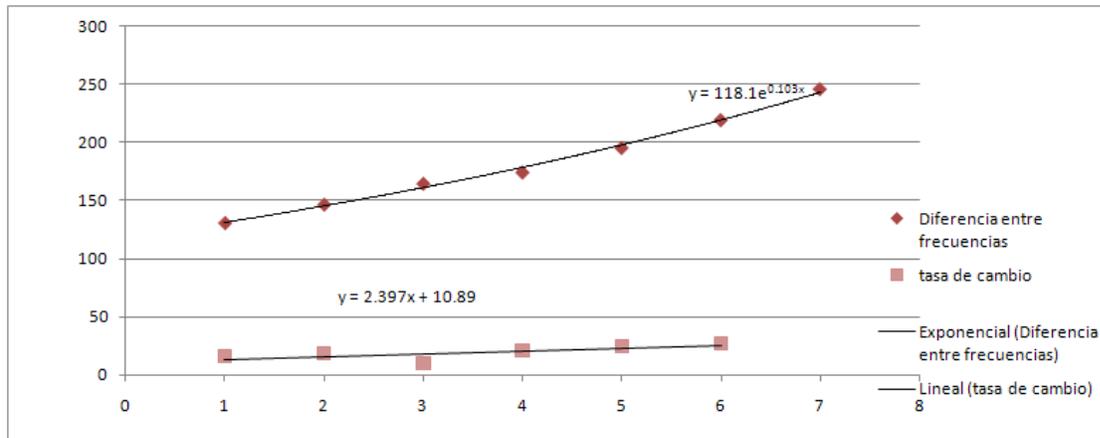
The chart above is a representation of what would be a C major scale and a G major scale. Given distances are a "perfect fifth" and the exchange rate between them is represented as follows:

### Diferencia entre las frecuencias

x	y
1	130.4
2	146.3
3	164.3
4	174.1
5	195.3
6	219.3
7	246.1

### Tasa de cambio

x	y
15.9	1
18	2
9.8	3
21.2	4
24	5
26.8	6



As scales advance, separation between fifths grows. This phenomenon is represented by the dark red spots, representing the distance between the notes (comes from the subtraction of the highest frequency to the lowest). The faded red dots represent its growth rate, which is obtained by subtracting the second note from the first and so on.

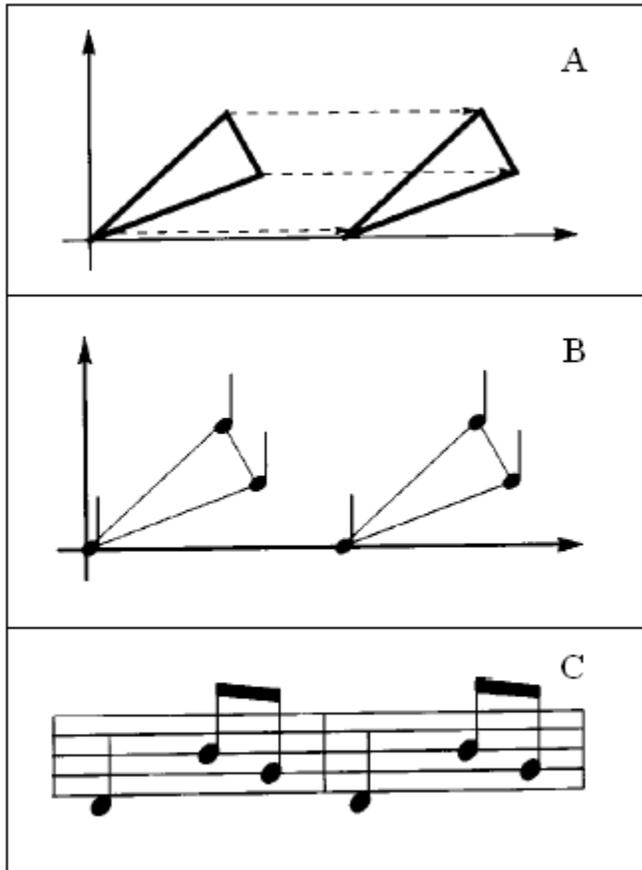
Some authors have tried to establish geometric relationships and although the ideas seem interesting, the concepts used do not seem to be the most accurate. This leads to multiple errors. In some ways (taking this assumption) we reach results that are out of context. Part of this paper discusses these nonsensical ideas.

Furthermore dodecaphonic and integral serialism use values they assign to a given Set.



This was an example of a twelve tone series, where the attributed values are typically 0 to 11 or in some instances from 1 to 12. Assigning a number to a frequency, rhythm and dynamics a 12 digits numbering assumes that the behavior is linear. In other words the values of the series are a line  $y = x$  aspect that would be a mistake. This does not mean it is wrong, it is a technique, but it would be wrong if we use such a model to represent and analyze musical structures, because the frequencies grow exponentially and in cases like the above series, that has a decreasing trend, it becomes somewhat contradictory. In Susana Tiburcio's thesis, referring to the work of Julio Estrada and Jorge Gil, melodic structures approach on a plane in order to analyze translations

and other behaviors. The following case D4, B4 and G4, one quarter note and two eighth notes. In the plane represented, D is placed at the origin, which will be the pair (0,0) so it is a complete nonsense to use that coordinate value. It may be interesting to see it as a representation, but this explanation does not provide us with any information.



The triangular shapes are meaningless. They will have sense as a melodic analysis where all triangles form 90 degree angles, and the other ones may vary depending on displacement.

In addition, in the musical structures there are many techniques that can be represented mathematically and that are easily represented if we use a numbering of note values similar to dodecaphonic system.



Scales can go up and down. And the intervals between the notes change, if the composers uses different modes.

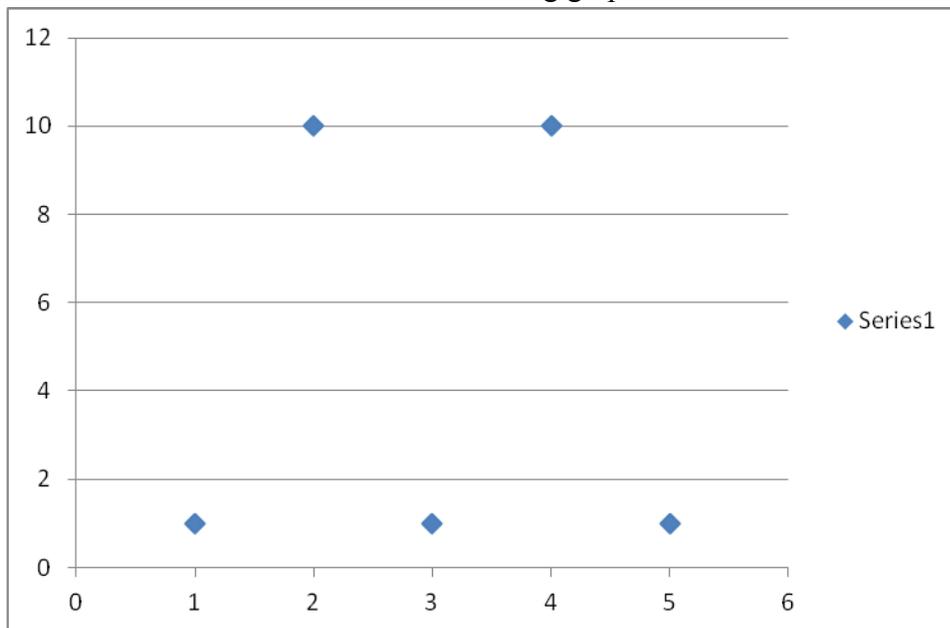
Also you can see how there is some relationship in numerical sequences. You can see that the

equation  $N = (p + 1)i + \left(9 \left| \sin\left(\frac{p\pi}{2}\right) \right| + 1\right)j$  for  $p = (1,2,3,4,5)$  results in the ordered pairs (1,10), (2,1), (3,10), (4,1), (5,10).



The equation and the values might vary from 1 to 12, which would be the 12 frequencies in octave 4. It can also be used, 1 to 7 numbering for the heptatonic scale. To obtain real results will be more convenient to use the frequencies, and approximate values of intensity. In music always two variables are manipulated more than the third (dynamics).

Their behavior can be seen in the following graph:



The equation can be varied in order to match the frequencies of the audible spectrum. To do this we modify the right side of the equation, we substitute 1 for 261.6 and 9 the difference between 440 and 261.6 (178.4) so, when the sine number is 1, both values will add up and the result of both will be A4 (440Hz).

$$N = (p + 1) i + (178.4 \left| \sin\left(\frac{p\pi}{2}\right) \right| + 261.6) j.$$

In this way each time you evaluate  $p$ , results will be 0 or 1 and the values associated with the variable  $f$  will be 261.6Hz and 440Hz.

The above case raises a very interesting aspect, in the case where  $p = 0$  then the function returns, the frequency 261.6Hz. The note should be extended and remain the same until the next note.

Hence the importance of introducing the concept of floor function with applied to a parametric equation. This in order to be more accurate in what the composer does introduce an additional concept we will see below.

## Floor function

In mathematics there is the case of integer function, floor. Something very similar happens in the perspective of the musician. From one note to another, this extends to the next value. In the same way as with the floor function.

We take vector  $(x, y, z)$  where:

$$x = T(p)$$

$$y = f(p)$$

$$z = I(p)$$

$T(p)$ ,  $f(p)$  and  $I(p)$  are functions

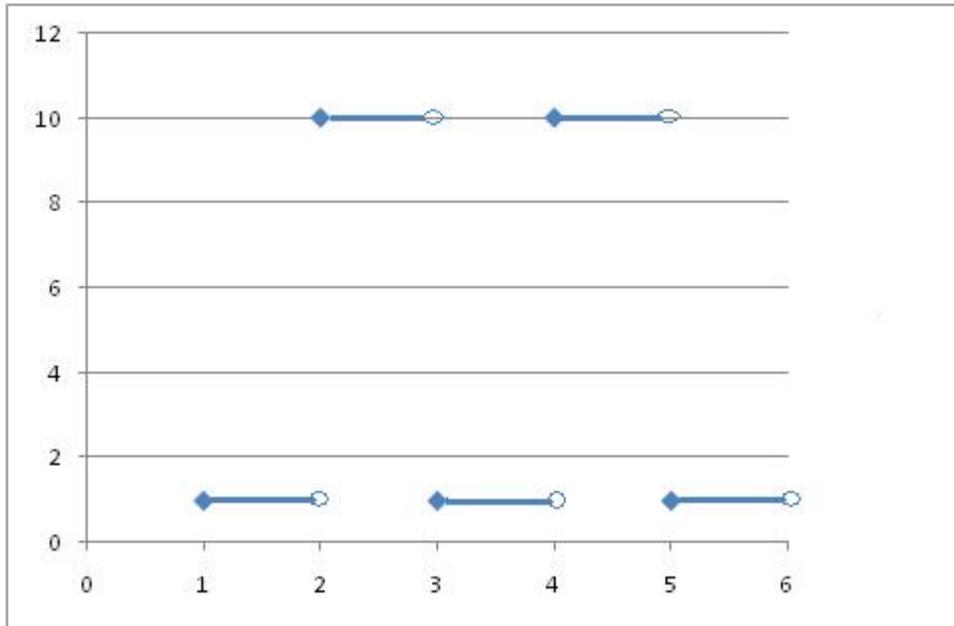
$$T(p), t \in R$$

$$f(p) = \begin{cases} f(p) & \text{si } t \in Z \\ \lfloor T(p) \rfloor & \text{si } t \notin Z \end{cases}$$

Then the function  $f(n) = (p + 1)i + \left(178.4 \left| \sin \frac{p\pi}{2} \right| + 261.6\right)j$  becomes:

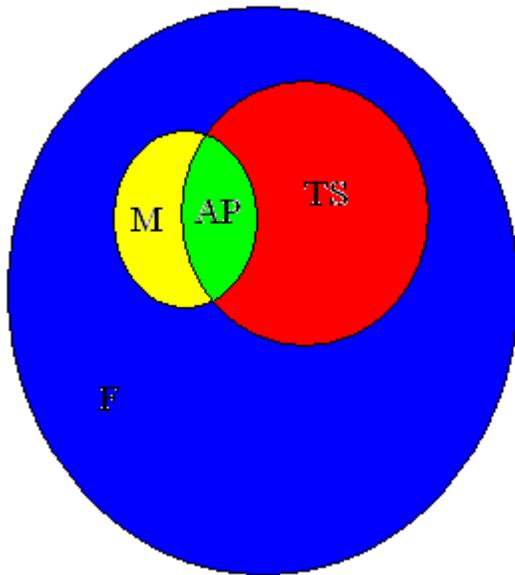
$$f(p) = \begin{cases} (p + 1) & \text{si } t \in Z \\ \left\lfloor \left(178.4 \left| \sin \frac{p\pi}{2} \right| + 261.6\right) \right\rfloor & \text{si } t \notin Z \end{cases}$$

Thus for each value of  $p$  then the function  $f(p)$  gives the result we want, even if it is a curve approximated by interpolation or some other method of polynomial approximation. For all other values contained between a point  $a$  and the other  $b$  the function which is the floor integer function.



## PERMUTATIONS

Permutations are ways of sorting the elements of a set or subset



The above drawing represents what would be the set of frequencies for both tempered tuning instruments, microtonal and others. Where F is the total set of frequencies, TS is the set of all octaves, including tones and semitones and M is the microtones, less used. The intersection of the set TS and M represented in green, and called AP, will be the acoustically possible frequencies.

Both M and its intersection TS are subsets of F.

F would be the entire audible spectrum.

The values shown in the frequency table are usually the most used. The microtones are not used so much and usually are accompanied by notes of the TS group.

The TS set contains all heptatonic scales, pentatonic, whole tone, semitone, twelve-tone series.

The whole microtonal set contains quarter tones and all possible tone divisions.

This makes it possible for a variety of motifs to exist, with their respective developments. So the permutation allows variability in musical language, as in formal languages. A property of a finite set is that its elements are able to be exchanged and, languages being finite sets of signs, permuted as required.

In the case of music, whether pentatonic scales, heptatonic, twelve-tone series, or any other form of scale, their elements permute. In the case of the pentatonic or 5 elements music subset, they can be exchanged 120 times in each octave, regardless of repeated notes. The widely used heptatonic scale has 5040 possible combinations in each octave and twelve-tone series for example has 479,001,600 ( four hundred seventy nine thousand, one thousand six hundred) ways to combine its elements for each octave.

One misconception is to believe that permutation of D4 and C4 are the same as swapping D5 and C5. Traditional music generalizes certain aspects of music notation to make them easier to

understand (like articulations); this works for the composer but does not help in understanding the music in a more accurate way. The fact that C4 and D4 and C5 and D5 keep a "numerical"  $2n$  relation, (refer to Table 1) does not mean that these are the same note.

Dodecaponic system uses groups of 12 elements, ranging from 0 to 11, assuming that 36.71 Hz (D1) and 73.42 Hz (D2) are the same note. This is evident in the harmonic rules that do not discriminate on the chord display through octaves if it is given the same treatment.

Chorales usually tend to have fewer breaks and the melodic curve is more homogeneous.

Obviously in the twentieth century these aspects change.

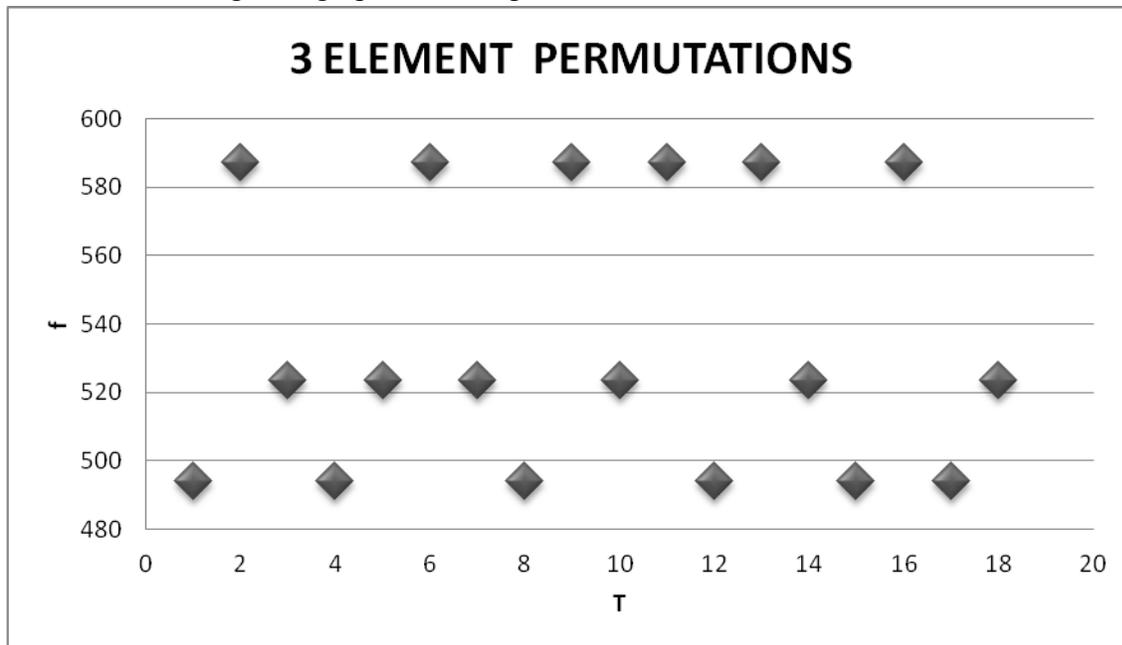
The next stave shows the possible combinations of three elements permutation



The frequencies associated are:

B 493.9Hz, C 523.3Hz and D 587.3Hz

These notes seeing on a graph show all possible forms in which each set of 3 vectors, can adopt



The chart above is representative of a subset of  $f$ , which includes 3 elements. If we repeat the same notes B-B-C, C-C-D and any other repeated elements, plus the same note repeated 3 times, then we have more combinations which also arise from permutating similar elements.

In the following example we take a set of seven elements and these elements are grouped into two. In a musical composition all combinations of two different elements that can be chosen are shown in the table, but there are more combinations, if we consider the same element to repeat.

The items listed below are G, A, B, C, D, E, F# represented in octave 4.

The notes are in the scale of G major and their associated frequencies are:

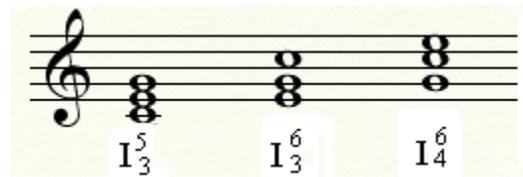


## VERTICAL PERMUTATIONS (Harmony)

So far we saw note displacements horizontally forming melodies, but permutations are also formed in chords. These are vertical relationships between notes and occur when two or more voices interact. The chords do not necessarily result in harmony, as music can be composed of a melody accompanied by chords. These chords can change the position of their elements, which is what musicians called inversions.

In classical music there is a tendency to use inversions. Fundamental in the bass, 3rd in the bass, 5th in the bass. This is a way to arrange the elements vertically as stated. Similarly each overlapping results in possible note permutations. A triad can swap 6 times. The root in the bass can be below a 3rd (E) and a 5th (G) or may be formed with the root under the 5th and the upper note will be the 3rd, in case of the triad. But there are differences when it comes to chords with a 7th, 9th, 11th, 13th.

In the example below we reverse the chord forms, with three of its permutations. First with the root in the bass, then with the 3rd in the bass and finally the 5th in the bass.



These distributions would be a C major chord. The numbers represent the intervals that are created between the bass note and the following ones. In the first case we have a 3rd and a 5th, in the second chord (first inversion) forms a 3rd and then a 6th over the bass note. The last chord and its distribution are a 4th and then a 6th. We conclude that chord inversions are vertical permutations.

In conclusion, all elements of a group are able to permute and music consists of a set of frequencies, a set of dynamics and a set of rhythms. These permute through a music composition.

To better illustrate the model we have:

T = time position

f = frequency

I = sound intensity.

If we omit the intensity in order to achieve more simple results then the generator set of our space is of the form:

$$T \begin{matrix} 1 \\ 0 \end{matrix} + f \begin{matrix} 0 \\ 1 \end{matrix}$$

The general form for three variables is as follows:

$$\begin{matrix} 1 & 0 & 0 \\ T0 + f1 + I0 \\ 0 & 0 & 1 \end{matrix}$$

## MELODIC CURVE AND POLYNOMIAL APPROXIMATION

Polynomial interpolation is a tool to approximate points on a curve. This technique is most often used in data but for purposes of this essay it can be helpful because it can approximate values to display the melodic curve. This also allows to approximate heptatonic scales, pentatonic and some melodies.

There are different types, of polynomial approximation, Lagrange, Hermite or cubic splines are ways to approximate the results, among others.

All methods have their advantages and disadvantages. It depends on what is required.

In counterpoint analysis, we study the structure of the melodic curve. This, as the name suggests, behaves like a curve. To represent it by a mathematical equation we must use short segments or it becomes complex and difficult to align. It is more useful and practical to try only on short segments. Such segments could be a phrase fragment or a motif.

In this way approximation can be achieved by using polynomials.

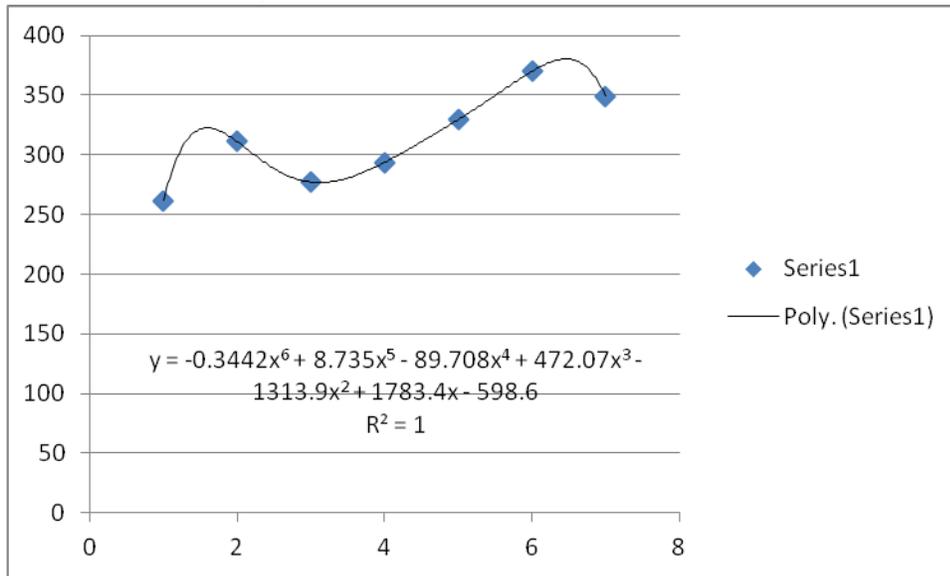
Another important aspect to stress is that the series of points approximated in terms of the frequencies must be close in terms of the parameter  $p = Z$  and each  $n$ , each  $p$  results are the positions in the axis  $T(p)$ ,  $f(p)$ ,  $I(p)$ . It is easier to represent the curve in its 3 variables.

Suppose a melody behaves as follows.

1	261.6	C4
5	311.1	D#4
7	277.2	C#4
9	293.7	D#4
10	329.6	E4
15	370	F#4
20	349.2	F4

Each rhythmic value is independent of the frequency behavior. So that the polynomial approximation for the frequencies is given in terms of values for  $p = Z$ .

And curve whose equation is:



The set of frequencies was built for values of  $p = (1,2,3,4,5,6,7)$ . In this way we obtain an equation that describes the behavior of each of the variables. Of course this would be a monumental equation but it is still a possibility. then:

$$N = T(p) i + f(p) j + I(p) k.$$

Whose behaviors are:

$$T(p) = 1 + \frac{9p}{4}$$

$$f(p) = | -0.344p^6 + 8.735p^5 - 89.70p^4 + 472.0p^3 - 1313p^2 + 1783p - 598.6 |$$

$$I(p) = 10 + 10p$$

defined by  $p = (1,2,3,4,5,6,7)$

The complete equation will have the form of:

$$N(p) = \left(1 + \frac{9p}{4}\right)i + (| -0.344p^6 + 8.735p^5 - 89.70p^4 + 472.0p^3 - 1313p^2 + 1783p - 598.6 |)j + (10 + 10p)_k$$

For frequencies are put in absolute value, in case you want to use other values in the equation and that are not present for the range in x of the function, which could generate negative values or negative frequencies which do not exist.

Values for each p will be represented as follows:

p	T	F	I
1	3.25	261.6	20

2	5.5	311.1	30
3	7.75	277.2	40
4	10	293.7	50
5	12.25	329.6	60
6	14.5	370	70
7	16.75	349.2	80

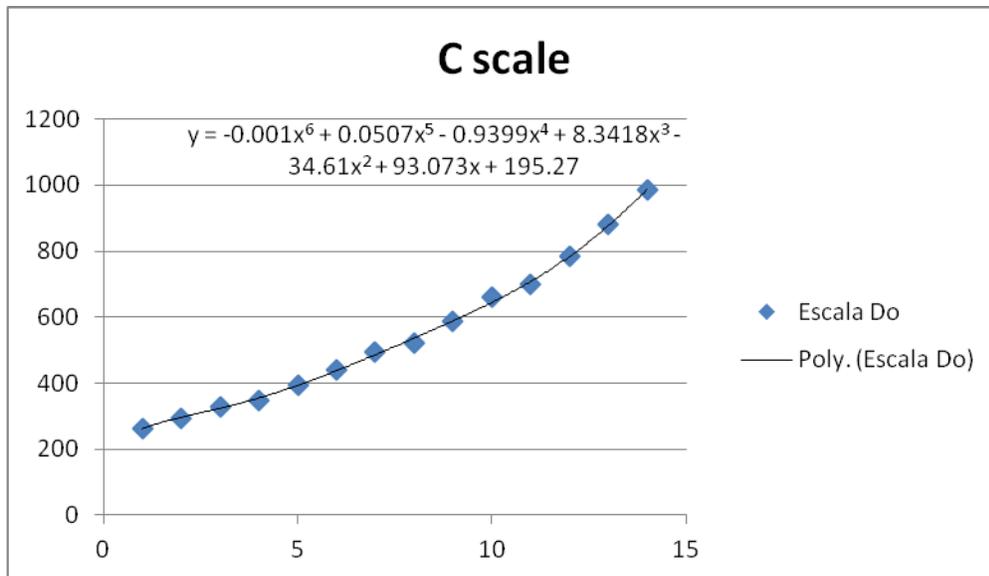
In practical terms, the use of these models certainly becomes a slow and tedious process if you wish to compose with this system but it is an interesting way to represent the model in a three-dimensional perspective for teaching purposes. Even if the values for the frequency shift is obtained with an equation of the form  $y = f(x)$ , can be parameterized by replacing  $x$  by  $p$ , then the resulting equation can be replaced in the following equation  $N = T(p) + (fp) + I(p)$ . The scale is a succession of notes ranging from C1 to C2.



Values in  $x$  and  $y$  are as follows:

1	261.6
2	293.7
3	329.6
4	349.2
5	392
6	440
7	493.9
8	523.3
9	587.3
10	659.3
11	698.5
12	784
13	880
14	987.8

And the chart is as follows:



The chart above represents an ascending C major scale that goes from a C1 up to B2, two equations are obtained using excel and both can be parameterized.

The result of evaluating the points  $x = (1,2,3,4,5,6,7,8,9,10,11,12,13,14)$  corresponds to specific values. Since the beginning we knew that all values generated by this equation go from 261.6Hz to 987.8Hz. This curve could be a glissando if it is seen as a continuous curve.

To apply an interpolation method to obtain a chromatic scale or a semitones scale would be crazy. In that case if the equation is known or can be obtained by using simple functions will be much easier than to interpolate.

In the case of the chromatic scale would be easier to do it this way:

We have an initial note where the scale begins and end. Then  $E = f_p \times 2^{\frac{p}{12}}$

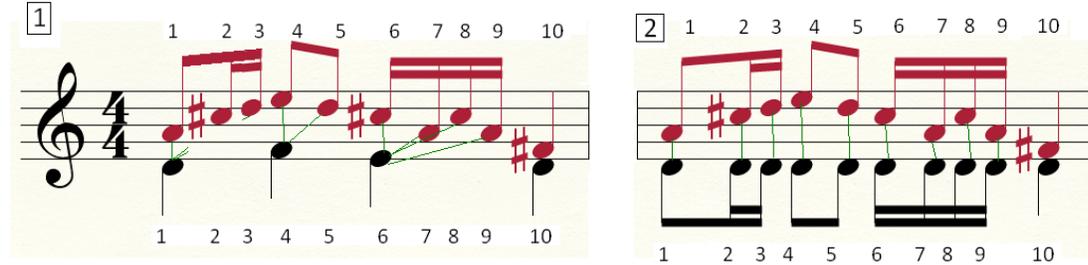
Where E refers to the scale, f is the initial frequency and the subscript refers to the number p for which it is evaluated and the exponential will be the ratio at which the frequencies grow,  $\sqrt[12]{2}$

Traditional music for instruments tends to be very sparse in terms of the values that are used, making it difficult to approximate certain motifs, or phrases. Usually its easier to approximate flatter segments which are not so scattered. To achieve greater accuracy is easier to determine a piecewise-defined function, to represent what you want, instead of a continuous one, to represent long segments of a music work. This will help to approximate a certain musical structure, with more accuracy.

In the cases that will be discussed later of Mozart and Bach, will be easier to see the points dispersion. Equations of this type and their graphic shapes would be interesting to use in order to generate musical structures, but keeping certain parameters.

# ORCHESTRAL MASS

When two or more voices interact with one another, relationships are created between the notes that interact. The figure below shows this phenomenon:



In the first bar, bass voice interprets the quarter notes. While the upper voice has a more dynamic rhythm.. The important thing about this example is that the upper voice notes for the first, second and third rhythmic figure are a combination of eighth notes and sixteenth notes, lower notes exert their effect on these. D4-A5 will be the first interval, or a perfect fifth interval. That lower voice D also exerts its effect on the notes C#5 and D5. and is where the note despite their frequency sounds takes effect immediately on the other notes that occur during its duration.

The statistics are an important tool that can help us understand the data and define important aspects. What happens if I take a musical segment using all values for that segment and introduce them on a graph. Using two bars of Bach's thirteenth invention:



In the above musical fragment it is important to remember two important aspects.

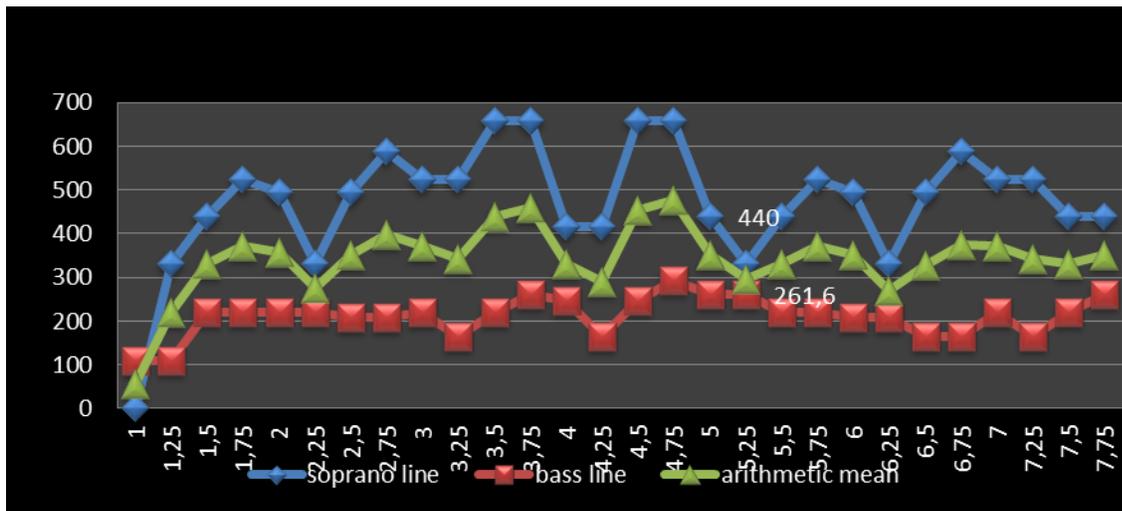
1) when two or more voices interact, if a note is long and the notes of the other voice interact simultaneously, but are shorter, intervals are created, for each one of the short notes. In the extract of invention XIII by Bach, the second note of the lower voice interacts with 4 top notes. The relationships that are created are as if instead of being a quarter note were sixteenth notes. It is not only are the notes that match which form a interval or a harmony.

Consider the following table:

T	Bass line	Soprano line	Distance between lines	Growth rate	x (growth rate)
1	110	0	110	0	1
1.25	110	329.6	219.6	109.6	2
1.5	220	440	220	0.4	3
1.75	220	523.3	303.3	83.3	4
2	220	493.9	273.9	29.4	5
2.25	220	329.6	109.6	164.3	6
2.5	207.7	493.9	286.2	176.6	7
2.75	207.7	587.3	379.6	93.4	8
3	220	523.3	303.3	76.3	9
3.25	164.8	523.3	358.5	55.2	10
3.5	220	659.3	439.3	80.8	11
3.75	261.6	659.3	397.7	41.6	12
4	246.9	415.3	168.4	229.3	13
4.25	164.8	415.3	250.5	82.1	14
4.5	246.9	659.3	412.4	161.9	15
4.75	293.7	659.3	365.6	46.8	16
5	261.6	440	178.4	187.2	17
5.25	261.6	329.6	68	110.4	18
5.5	220	440	220	152	19
5.75	220	523.3	303.3	83.3	20
6	207.7	493.9	286.2	17.1	21
6.25	207.7	329.6	121.9	164.3	22
6.5	164.8	493.9	329.1	207.2	23
6.75	164.8	587.3	422.5	93.4	24
7	220	523.3	303.3	119.2	25
7.25	164.8	523.3	358.5	55.2	26
7.5	220	440	220	138.5	27
7.75	261.6	440	178.4	41.6	28

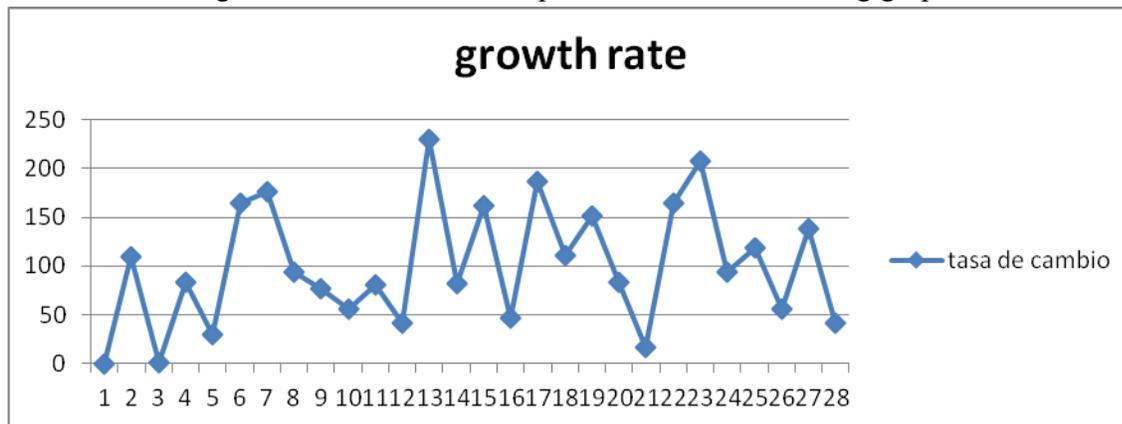
A T value represents the position of each note. As mentioned above, the second note in the bass voice (quarter) affects 4 of the top notes (eight notes). That note produces different intervals with each of the elements of the upper melody. So, it is not analyzed as a quarter note but as a sixteenth note which forms a relationship note to note. Point against point. The second two columns are the frequencies of each of the voices. The fourth column is obtained by subtracting the voice frequencies of the soprano voice to the bass voice. The growth rate is obtained by subtracting  $f_{p+1} - f_p$  which is the distance between one frequency and another. The sixth

column gives the differences between each of distances, which would be the change that occurs between intervals.



The lower melody represented in the graph with a red line tends to be a little more linear and less steep than the one represented in blue. So the arithmetic mean (green) runs through the middle of the other two. Where there are two voices the arithmetic mean will be more pronounced for the jumps of voices. Where there are more voices, average behavior tends to be both linear and without abrupt jump. Now this is a case that includes only two voices. The larger the orchestra is the average line becomes much more linear without many abrupt changes. Duplications favor this.

As said the change between intervals is represented in the following graph:



It is known that in traditional music C and its octave, represents a relation exactly twice,  $2^n$ .

If we multiply the difference between a note and the next by a factor  $2^n$  we get an interesting result. For example between any C and C# through octaves, exists an exchange factor of  $1.95 \times 2^n$  between both.

Between C and G we see that this same relationship holds with a rate of change that would be:  $16,3 \times 2^n$ . This implies that the growth rate between any two notes which are closest in frequency is shown by the difference between the two frequencies, multiplied by the factor  $2^n$ .

Music is like a set of permutations on the same scale, in its modes or in other scales. In musical analysis of various composers is quite interesting to see how the melody behaves. This phenomenon is quite usual in Bach. But not only in his music, also in many other composers. In the orchestral works, many composers duplicate certain notes from the chord. This can be from a 5 part orchestra to something bigger, where the number of times a note duplicates may vary according to musical period.

Clearly this depends on culture and the time when the composer lived. This orchestral mass, moves and we could say that if we take the arithmetic mean of the overlapping frequencies, we can get a orchestral mass center of harmony. As in physics, there is center of mass of a particle or particle group or a body, we can see the notes as multiple particles placed above one another creating a center in which they move through the whole music piece. In some composers or in some music works this orchestral mass center tends to be more flat and uniform, others may vary. So this orchestral center of mass will tend to be more uniform in a baroque composition than in a work of integral serialism, which not only has a tone serial process but also other musical components such as dynamics and rhythms.

## Minuet

(from the Opera "Don Giovanni")

Wolfgang Amadeus Mozart  
(1756-1791)

Moderato

mp

5

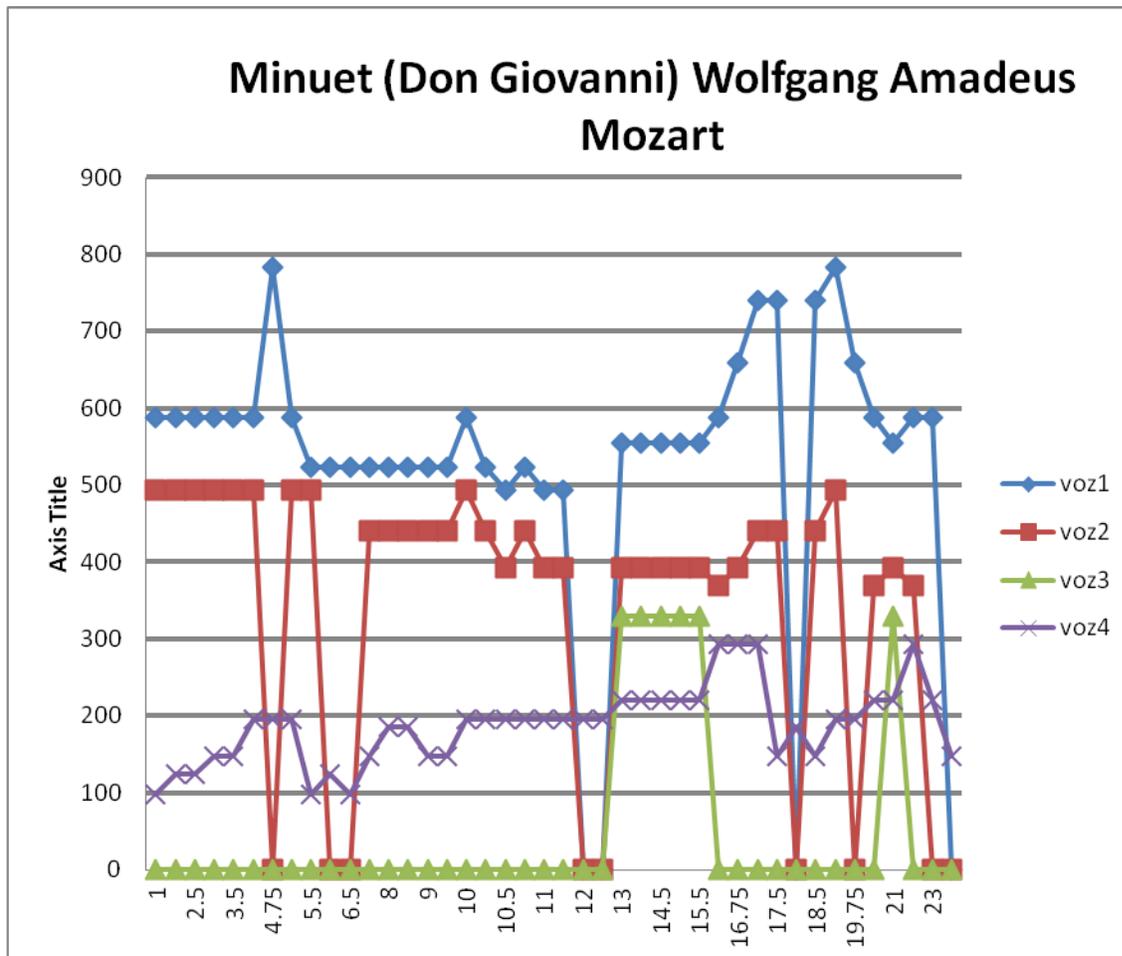
Fine

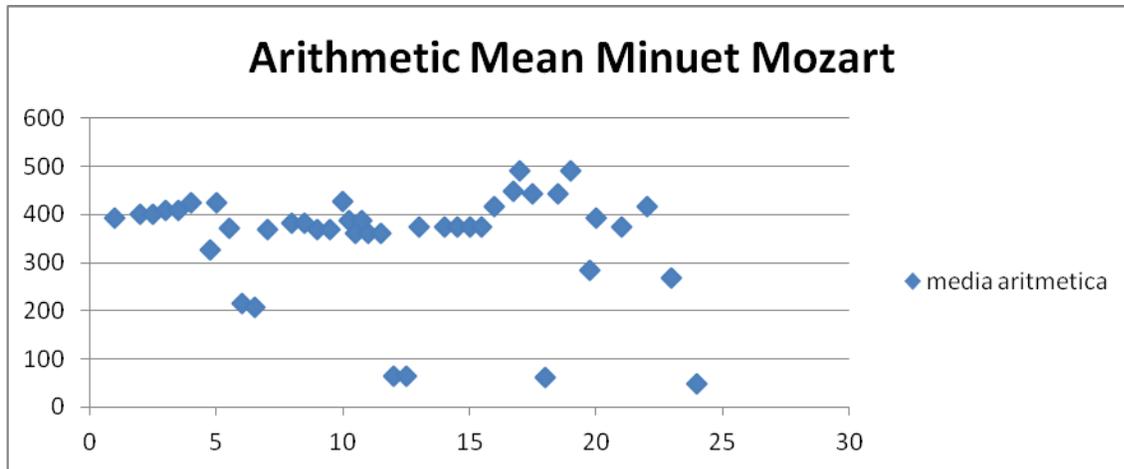
mf

The image shows a musical score for a Minuet by Wolfgang Amadeus Mozart. It is in 3/4 time and G major. The score is divided into two systems. The first system (measures 1-4) is marked 'Moderato' and 'mp'. The second system (measures 5-8) is marked 'Fine' and 'mf'. The score is written for piano, with a treble and bass clef. The first system shows a melody in the right hand and a bass line in the left hand. The second system shows a continuation of the melody and bass line, ending with a double bar line and repeat sign.

T(p)	soprano line	contralto line	tenor line	bass line	arithmetic mean
1	587.3	493.9	0	98	393.06
2	587.3	493.9	0	123.47	401.55
2.5	587.3	493.9	0	123.47	401.55
3	587.3	493.9	0	146.8	409.33
3.5	587.3	493.9	0	146.8	409.33
4	587.3	493.9	0	196	425.73
4.75	784	0	0	196	326.66
5	587.3	493.9	0	196	425.73
5.5	523.3	493.9	0	98	371.73
6	523.3	0	0	123.47	215.59
6.5	523.3	0	0	98	207.1
7	523.3	440	0	146.8	370.03
8	523.3	440	0	185	382.76
8.5	523.3	440	0	185	382.76
9	523.3	440	0	146.8	370.03
9.5	523.3	440	0	146.8	370.03
10	587.7	493.9	0	196	425.86
10.25	523.3	440	0	196	386.43
10.5	493.9	392	0	196	360.63
10.75	523.3	440	0	196	386.43
11	493.9	392	0	196	360.63
11.5	493.9	392	0	196	360.63
12	0	0	0	196	65.33
12.5	0	0	0	196	65.33
13	554.4	392	329.6	220	374
14	554.4	392	329.6	220	374
14.5	554.4	392	329.6	220	374
15	554.4	392	329.6	220	374
15.5	554.4	392	329.6	220	374
16	587.3	370	0	293.7	417
16.75	659.3	392	0	293.7	448.33
17	740	440	0	293.7	491.23
17.5	740	440	0	146.8	442.26
18	0	0	0	185	61.66
18.5	740	440	0	146.8	442.26
19	784	493.9	0	196	491.3

19.75	659.3	0	0	196	285.1
20	587.3	370	0	220	392.43
21	554.4	392	329.6	220	374
22	587.3	370	0	293.7	417
23	587.3	0	0	220	269.1
24	0	0	0	146.8	48.93

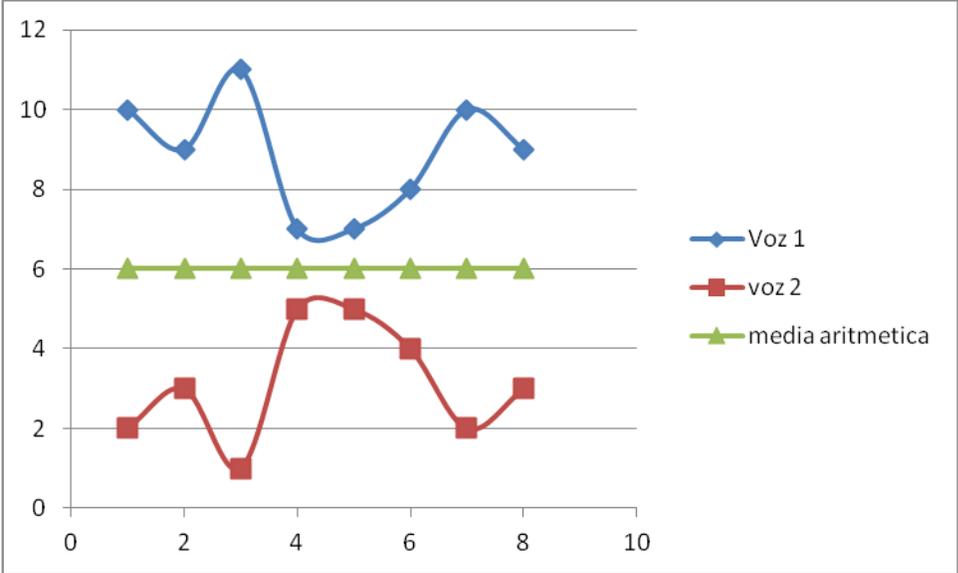




The above chart represents the arithmetic mean.

An interesting aspect would be to build a work where the arithmetic means stay constant, meaning that all the notes spin around it, without altering its value. This axis can be used with a frequency that works as a center of "orchestral mass" where other notes are build downside or upside of it, creating a perfect balance. In ancient choral music because of the symmetry of its displacements musical segments could be found that can resemble this idea, but they will not reflect this idea perfectly in every aspect. This is another detail that is presented but is no more than as a suggestion for the reader, in case someone would like to do a thorough analysis of such forms or to make a structure that meets the proposal. It should be as follows:

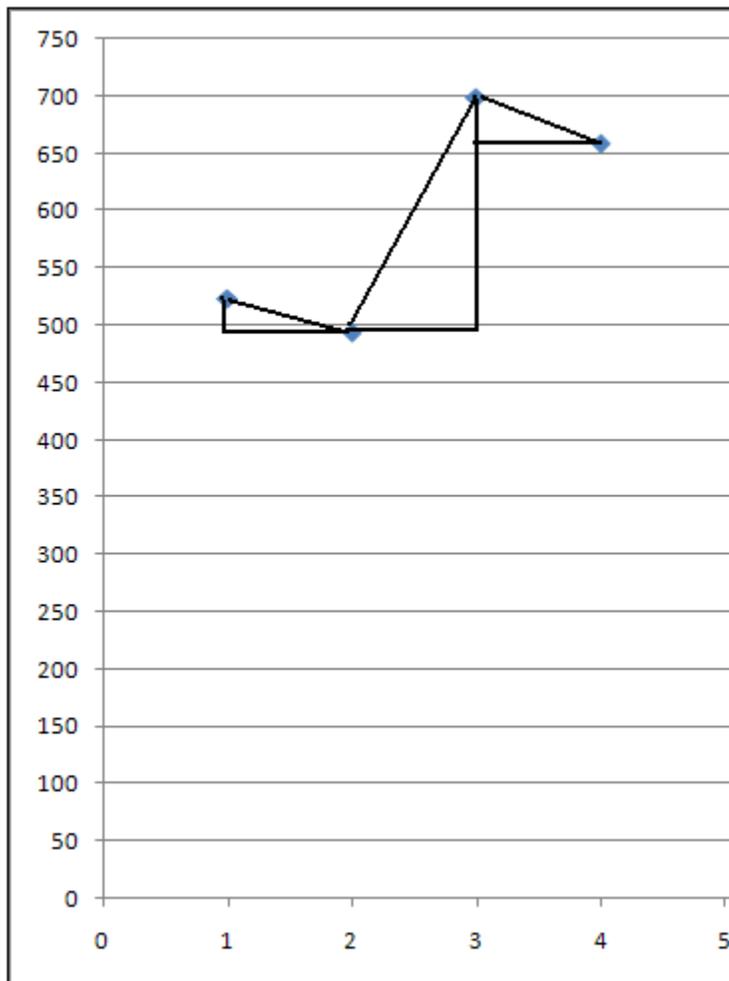
y1	y2	y3	x
2	10	6	1
3	9	6	2
1	11	6	3
5	7	6	4
5	7	6	5
4	8	6	6
2	10	6	7
3	9	6	8



# QUADRATIC THOUGHT

In instrumental music all melodic movement between voices can be summarized as intervals. Leaving aside the interest of highlighting the melody and considering that usually rhythm and tone are the two most dynamic variables.

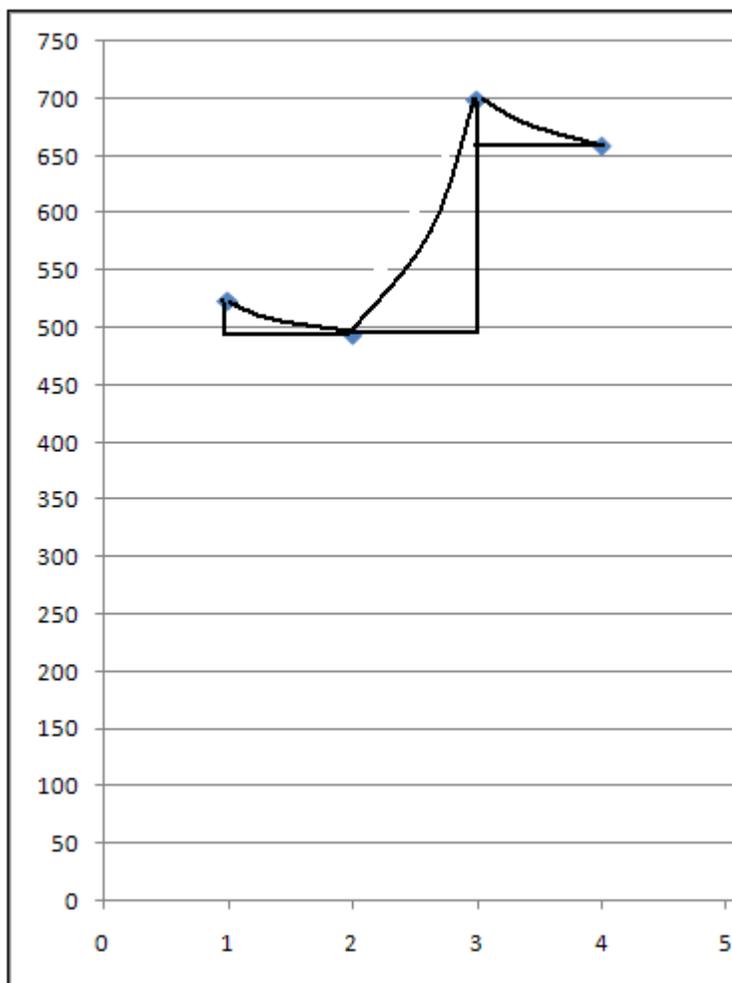
In this chapter we will see two cases, the case of two or more consecutive vertical intervals and intervals for consecutive horizontal (melody). We will start with melodic analysis.



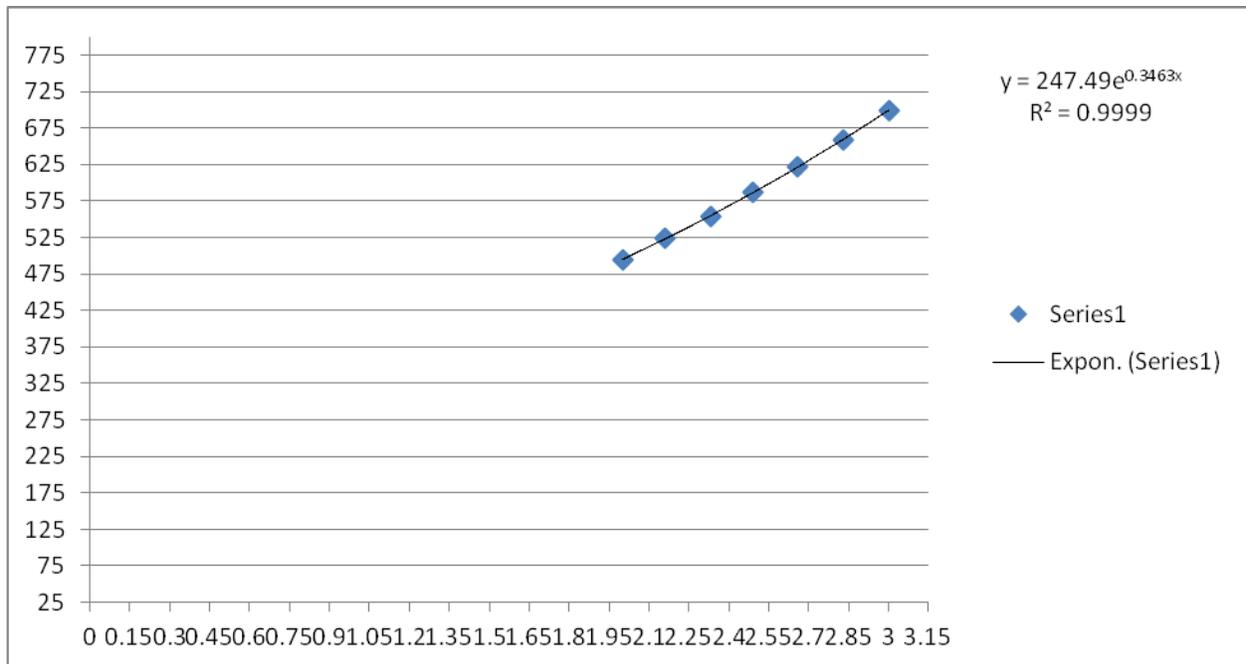
Whose values are:

1	523.3
2	493.9
3	698.5
4	659.3

These triangles and the Jorge Gil and Julio Estrada geometric proposal, certainly are interesting structures. But there is a problem, these structures are not triangles. Actually scales behavior is not linear, it is exponential: then it would be something more like this chart (even though the curves are exaggerated):



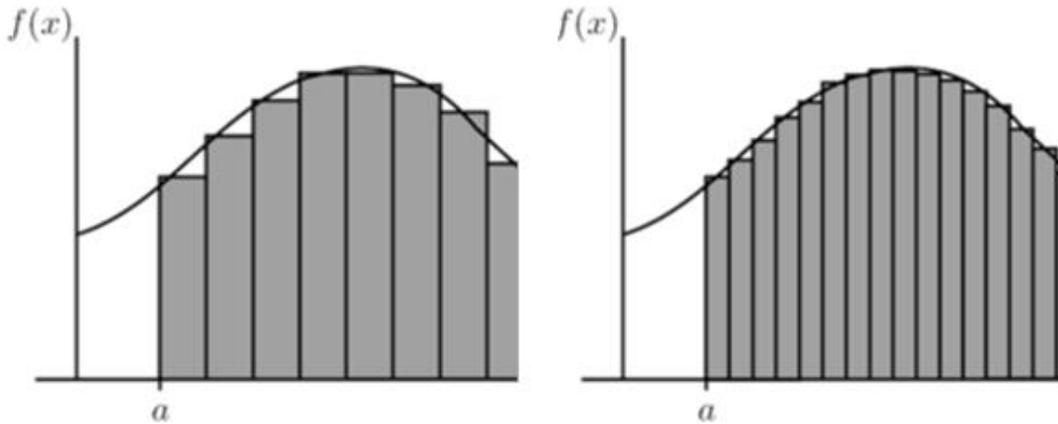
If we use straight lines, these lines do not provide any information and are a mistake because the exponential curve that forms between the notes contain the actual values, while a straight line does not. This is best seen in the following graph, which is a little more accurate and belongs to the second triangle that is formed.



The chart above is not as pronounced as the graphics that were before, but it is still not linear behavior. All values used were the number of frequencies that can fit in that augmented fourth interval.

2	493.9
2.16	523.3
2.33	554.4
2.49	587.3
2.66	622.3
2.83	659.3
3	698.5

In the analysis of harmonic blocks, squares to be discussed later are formed on any progression of 2 or more intervals. The parallelograms, present in the music of Mozart and other musical genres have a great importance. These genres and the modern popular music tend to repeat these structures a great deal. An analysis of music in these terms suggests that a real break from tradition can not be made with acoustic music, the real break will come with electroacoustic music which means the elimination of these triangles and squares. The square formed between these structures, using a computer can be minimized and would become like Riemann sums where the idea would be to minimize the size of the triangles in the melody and the squares in the harmony, and so obtain the desired result, continuous and not fragmented as tonal music is.



The greater the number of blocks, the smaller their size and the more accurate the result of the Riemann sums. This mathematical procedure does not apply in this essay but serves as an explanation to what it's been looking for. With the elimination of these blocks we can overcome an aspect that represents the tradition, from early music to this day.

If a program can be created capable of generating a surface (solid of revolution) much of the tradition will be overcome, it is an object that completely eliminates the triangles formed between consecutive notes (melodies) and the blocks (harmony) that are formed between consecutive intervals.

Let us see what these blocks are and what their origin is. Then we have 4 intervals.

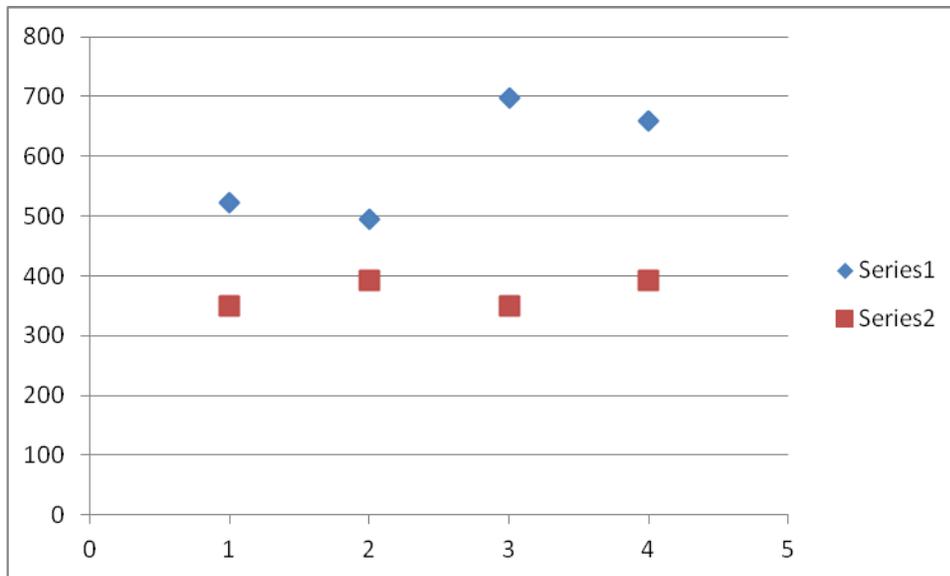
- 1) perfect fifth
- 2) major third
- 3) octave
- 4) major sixth



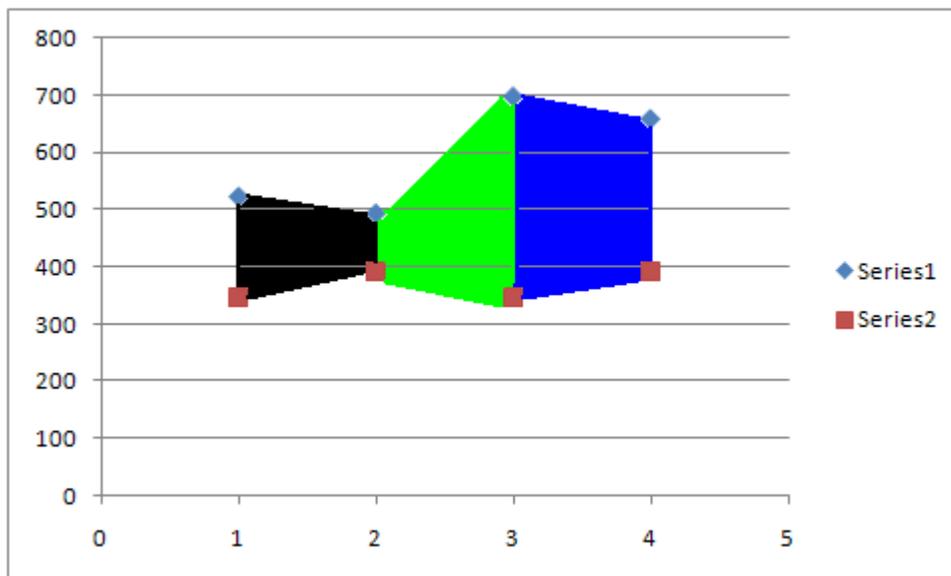
Associated values for both variables are:

1	523.3	349.2
2	493.9	392
3	698.5	349.2
4	659.3	392

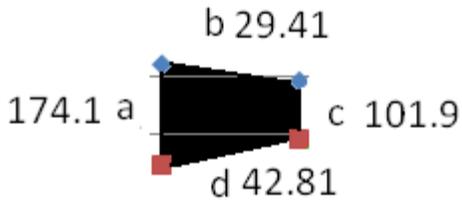
And the graph that forms is this:



The intervals are created between two consecutive notes and are quadrilaterals. If we use the colors to differentiate them, it is appreciated that 3 blocks are formed for that 4 interval progression. A color representation will be: first quadrilateral in black, the second in green and the third in blue.



Let us take the black quadrilateral to analyze it separately. Its sides are a, b, c, d whose respective values are 174.1Hz, 29.41, 101.9Hz, and 50.8. This is not a parallelogram, such would be the most common where two sides are parallel and the other two form an angle in relation to the horizontal.



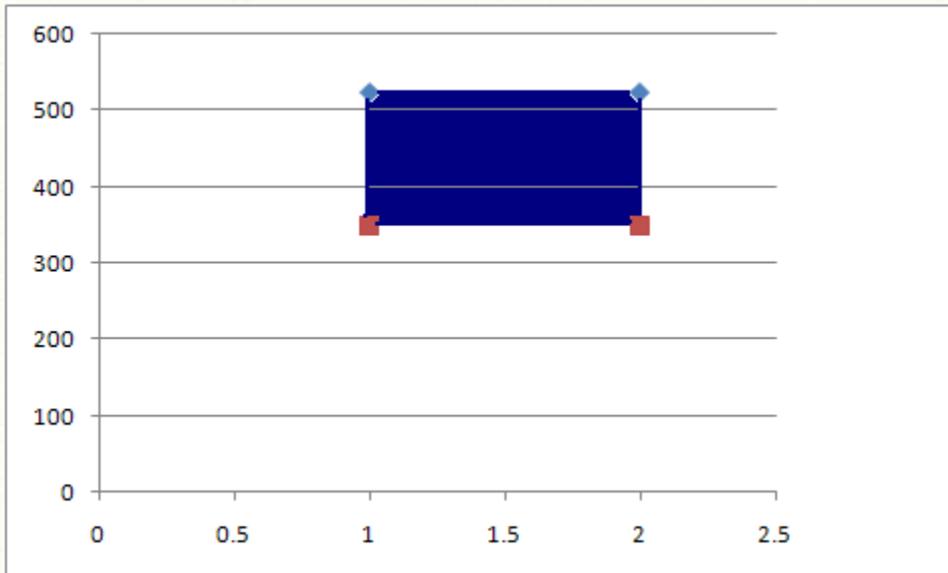
The distances in a and c are given in Hz, they will be 174.1Hz and 101.9Hz.

The above example using quadrilaterals also has a problem, that even if points are too close one to the other, lets say a minor second, then is not a linear behavior is an exponential behavior.

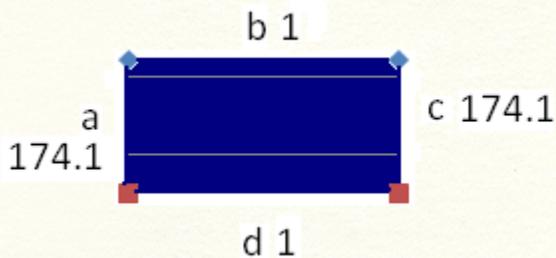
The only thing you need to get a parallelogram will be two consecutive and equal intervals.



graphically would be:



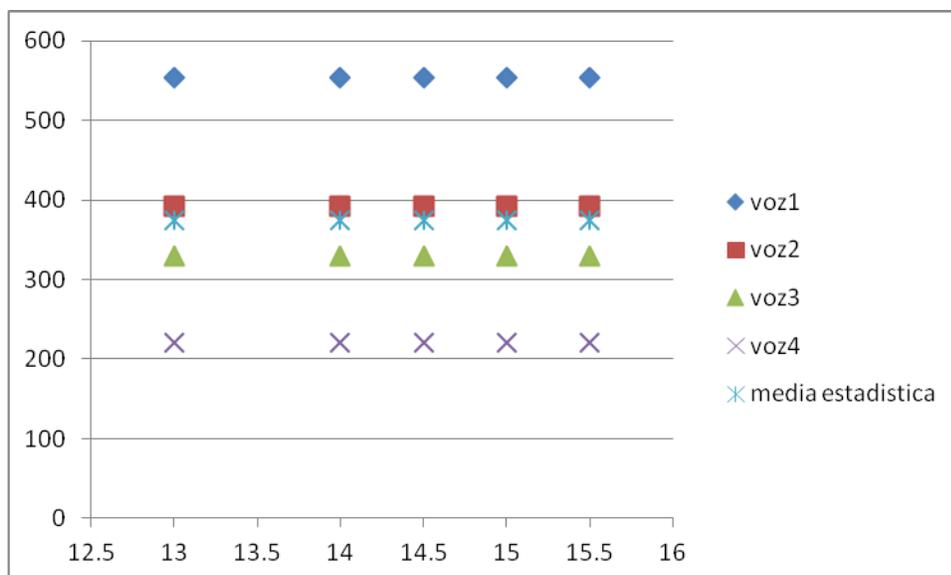
This is a parallelogram whose values are represented in the figure below:



These blocks are present in the example of the previous chapter, the case of Mozart's Minuet, which are quite common throughout the work. We can obtain from rhythm setting 13 the following data table.

13	554.4	392	329.6	220	374
14	554.4	392	329.6	220	374
14.5	554.4	392	329.6	220	374
15	554.4	392	329.6	220	374
15.5	554.4	392	329.6	220	374

The graph associated to the same table would be:



The A7 chord is repeated 5 times. The arithmetic mean would be the center where other notes are distributed. This type of behavior even though was present in Bach was not as common, while in the classic period became more usual. This is the same as in modern popular music. In rock these repeated sequences of chords are very common.

This is different from serialism, where dynamics are in constant change.

Integral serialism is a very particular case. It uses a series for the tones, similar to the twelve-tone series but it covers other aspects of music, such as rhythm, tone, instruments etc. Unlike other musical styles where a segment has variations in T and f, but I is held constant, in serialism this aspect changes. Each note is assigned a different intensity and ways in which it is distributed in the Cartesian plane are different.

The next series, includes frequency, rhythm and intensity for each given note.



Following the classical theory this can form a matrix with the Original, Inverted, Inverted Retrograde and Retrograde.

Similarly intervals are formed which also form blocks similar to those proposed in the previous examples. However, this technique is not constant in terms of loudness, so there are blocks whose sides are displaced in the Cartesian plane. It is quite interesting that, in linear algebra two vectors can be orthogonal but they do not intersect. They just form the angle but they can be located in any part of the space. In this case, unlike Mozart Minuet, the change in dynamics will make position vectors move from left to right on the z axis. If we ignore the Z axis these quadrilateral structures remain ordinary as previously described.

The composer selects and manipulates the notes as he desires in many styles, including integral serialism where the series is picked by the composer. In Xenakis' musical works where the notes are generated by mathematical algorithms or theories applied, the selection of notes is controlled by those. The case of UPIC is usually somewhat different, as the drawings are usually continuous lines. This form of composition is not so common, it does not use the traditional models; they are structures built from notes obtained from mathematical results that stay attached to their algorithms. In the case of UPIC the music created is a result of drawings.

Overcoming a tradition, is not achieved by giving dynamism to the axis I (Intensity), requires a delineation of all the aspects contain in tradition. Once established they can be eradicated one by one and this is achieved only with mathematical models for musical development.

This is the point where many may ask.

At which point would these new structures be something that we can call music?

It is subjective.

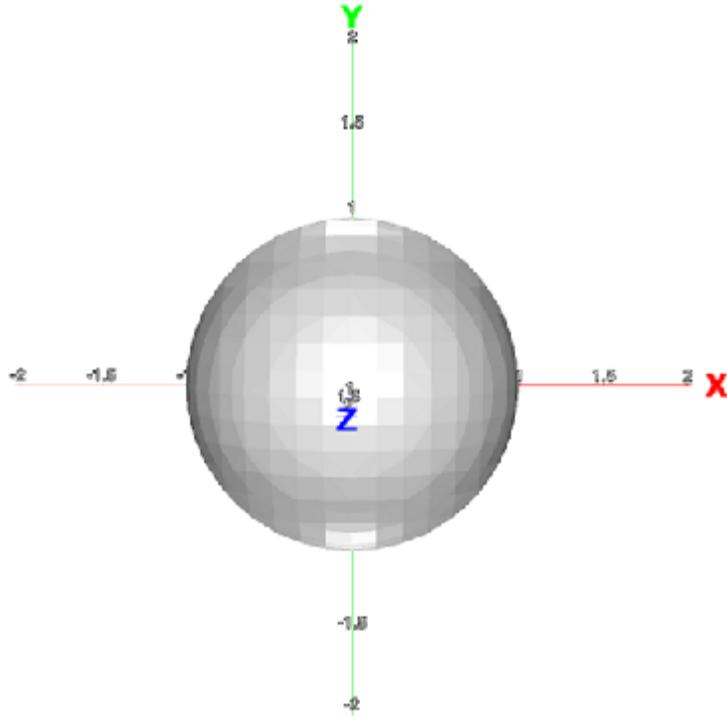
From the above we conclude that a model that breaks this scheme would be the use of continuous lines, rather than specific values, even if floor function is applied. Solids of revolution or surfaces in R3 can break this scheme.

In this way there is no formation of "squares" [*block?/quadrilaterals?*] or "triangles" and the displacement of the voices does not correspond to specific values, they will be continuous intervals where the frequencies are displaced without the traditional edges of acoustic music.

Revolving solids such as the sphere or other surfaces whose values are within the audible spectrum will be an option that would yield new and interesting results.

We use as an example the sphere centered at (0,0,0) of radius 1, whose equation is:

$$x^2 + y^2 + z^2 = 1$$



The above sphere is not valid due to its center and radius. A frequency of 1 Hz is not perceived. This is due to one side of the sphere being located from 0 Hz to 1 Hz, so, there will be no perception whatsoever. Now if we move the sphere toward a center where the radius has the ability to turn on the spectrum and loudness then, we can achieve relevant results for representation.

By the general formula of the equation with radius  $r$  with center at  $(a, b, c)$ ,

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

we can place an sphere and move it all to fit the positive axes, and center it at points that belong to the audible spectrum and intensity. This equation is represents the shell of the sphere, if we want to get the internal points then would occupy an inequality, which would be much more complex to achieve. If we see the sphere as a number of rings then we can build ring over ring, and then elaborate a sphere-shaped structure.

Using an inequality, then we have a variable radius, an aspect that includes all possible inner circles of the sphere, which are infinite, but can be delimited. A sphere alone is unable to establish two sounds that sound at the same time and a single sound at the beginning and end. Using internal spheres for each sphere you add two more frequencies which will interact with the shell. Each new sphere added provides one more sound at the beginning and end and two sounds during its displacement.

In the external sphere it forms 1 sound at the beginning of it, and then 3 frequencies interacting at the beginning and end of the inner sphere and then 4 frequencies interacting in its displacement. This works the same for each sphere included.

Suppose for a moment that using a radius which is located within the most common frequencies 27.5 Hz to 4186 Hz we can select a radius that extends through a range located inside those two values, lets say 60, so the sphere will have a diameter of 120. This is because we need to locate it before the pain threshold of hearing.

The center of the sphere is located on the frequency 311.1Hz or D# so it will be built only in the positive values that are needed.

Then suppose that z is centered at 80 decibels, and the radius ranges from 20 decibels, e.g. the loudness of a library, and the 140 decibels where the pain threshold begins.

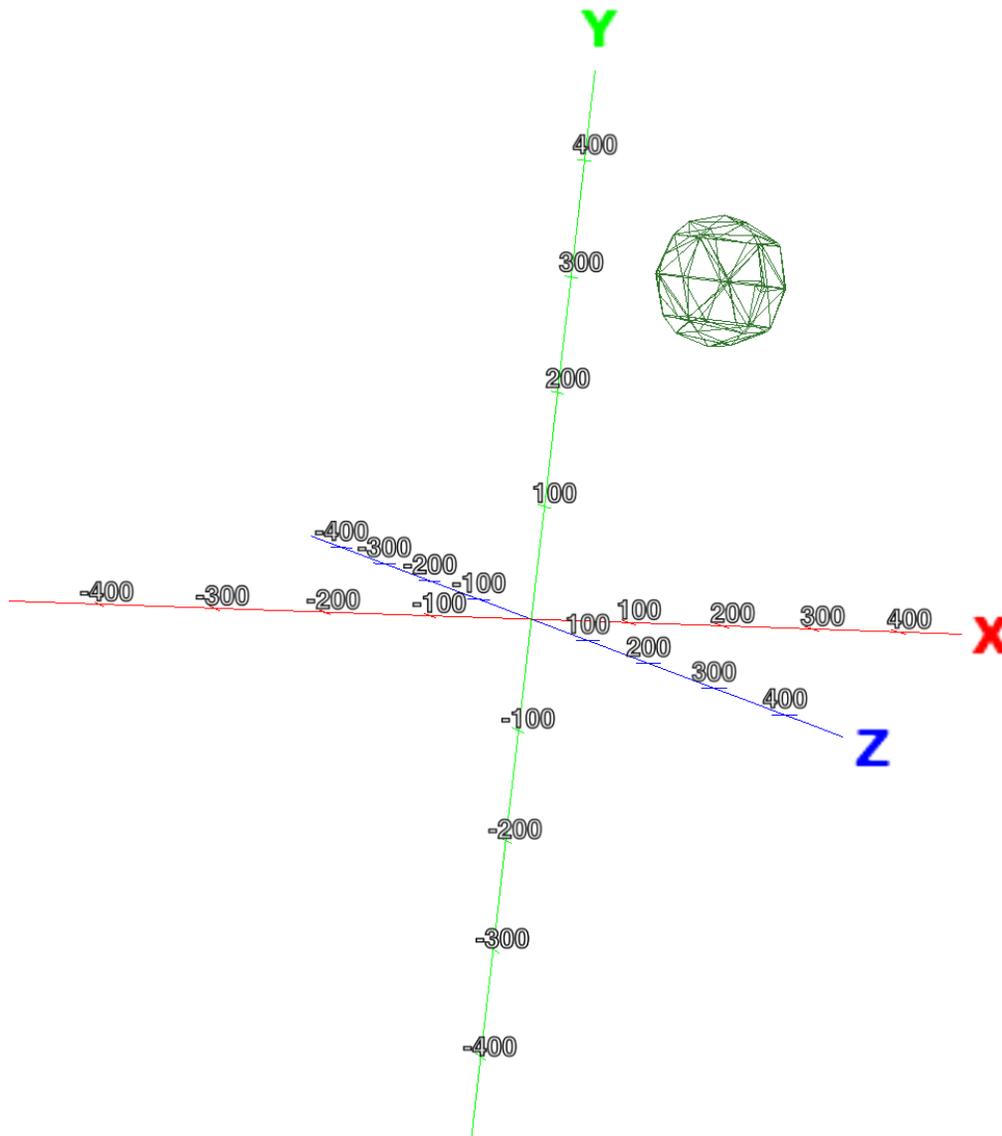
The center on the x axis is 100, so I can again contain only positive values and the displacement of the radius will never cross to the negative side of the axis.

With this selected values I have a sphere located entirely on the positive side of the axes and so I can generate a sphere that produces sounds within the audible spectrum and within tolerable levels of loudness.

The new equation of the sphere would be:

$$(x - 100)^2 + (y - 311.1)^2 + (z - 80)^2 = 60^2$$

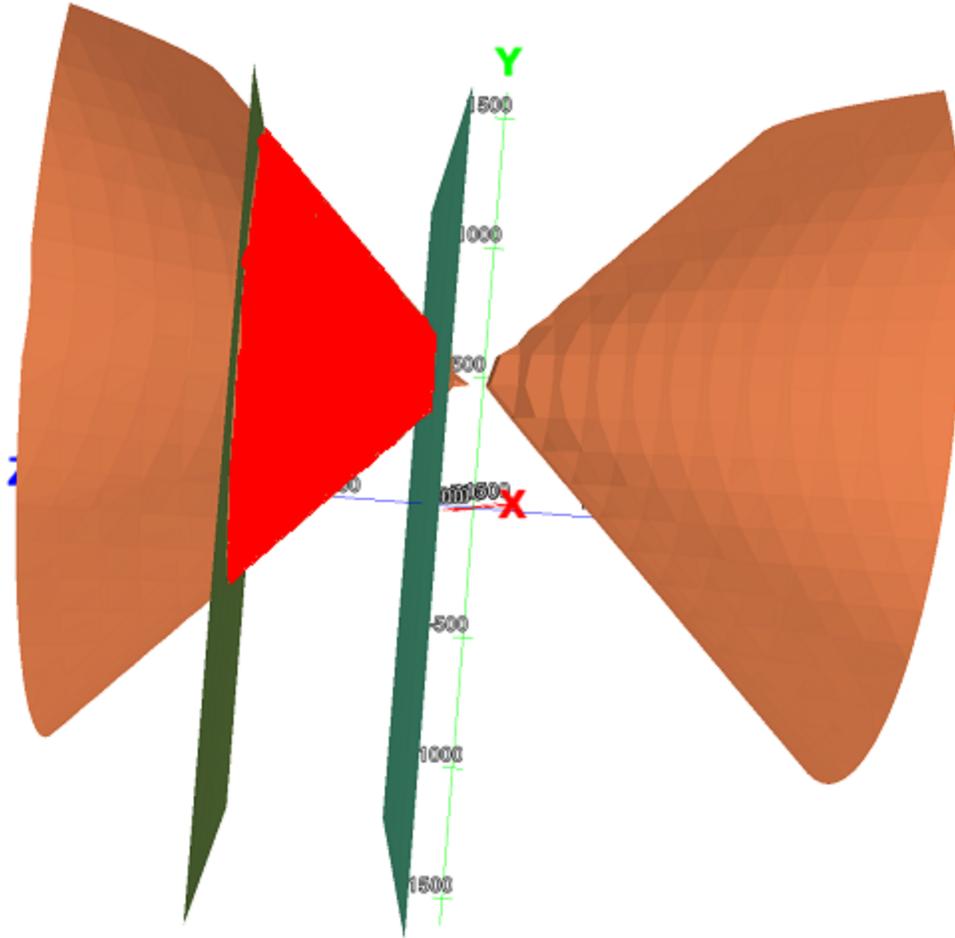
And the graph that represents it is of the form:



It is important to note that the area in order to be played in sounds must be, interpreted as two separate voices. A plane intersects the sphere over the y and z axes. Time moves continuously but the sphere has 2 sides and the occupy values can be placed separately.

In this way we obtain a sphere, but we can also obtain any other desired surface. For instance, paraboloid, hyperboloid, elliptic paraboloid and all others can be formed by manipulating their center and position in the plane to match a sound structure capable of being heard.

In a surface of the form:  $(x - 50)^2 + (y - 500)^2 - (z - 40)^2 = 30$  defined by  $z = 200$  to  $z = 900$ ,



the usable fragment its defined between the 2 planes shown in red.

Having analyzed the musical phenomenon from the vector point of view, an extension to the topic arises. Picking up some ideas in which music is ascribed certain properties that would be location, frequency and intensity, some composers say that mixing certain intervals or certain timbres have a, sweet, sharp, robust or other adjective to the produced sound.

However these are abstract aspects of each and every one perspective, and others may or may not share this classification.

The auditory system properties play a key role in how we capture the sound. The same happens with color, some animals have different ways of perceiving it, subject to its functioning color receptors. They can be dichromate, trichromate, tetrachromate.

The senses perceive and process the information received from the characteristics of the perceptive organ associated, and this information will be interpreted accordingly. Color and sound are waves, one electromagnetic and the other mechanic. The classification that a person should give to a particular sound or combinations of sounds remains a sound, and color is just a

color, mechanical waves and electromagnetic waves manipulated to produce what we call art. They are as we perceive them, and our brain interprets them as they are: they are just another interpretation of the external environment. So if we question the interpretation that perception gives on a perceived phenomena, the results and interpretation of experiments should also be questioned. Beyond the physical questioning about the world we should take a similar interest in understanding how the brain interprets the information. How the information received through the senses is processed, in order to be manipulated. This will be discussed in the next chapter. These type of patterns in Mozart's music are reported to have specific effects on the brain (referred to as the Mozart effect). David Solomons a contemporary composer from the UK reports that one of his songs (which is built upon comparable patterns) made a child who was autistic react in an emotional way (song called "My Love": <https://www.youtube.com/watch?v=71NPLYkq8W8>). Before that she had had no reactions to anything. Her parents were delighted that something could elicit an emotional reaction from her.

## IS ART THE KEY TO UNDERSTANDING THE STRUCTURE OF THOUGHT?

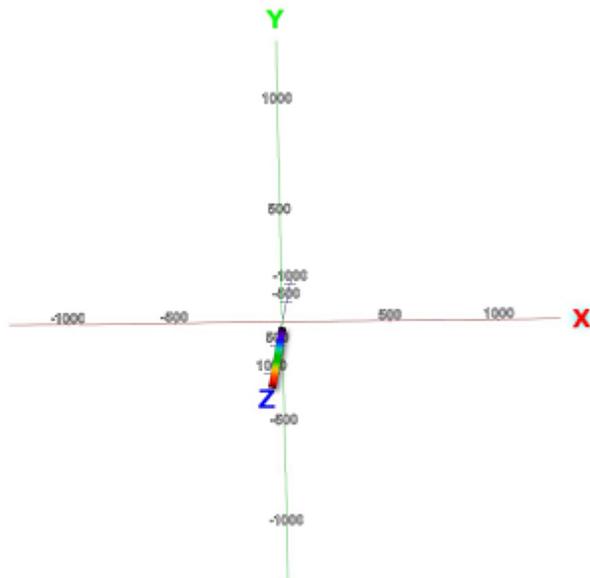
From the vector perspective as a basic unit capable of representing the musical note in its three variables, we can also extend this representation to other arts. Such would be the case of the visual arts. This are formed from two coordinates to place the points, strokes or brush strokes and has another that determines the color coordinate.

After analyzing the perspective of some painters, they refer to painting as the mixture of different variables, such as composition, brightness, tone, saturation, and others. The characteristics they mentioned are manipulations of the frequencies contained in the visible spectrum, which is nothing more than a wave whose frequency as in the case of mechanical waves can be delimited.

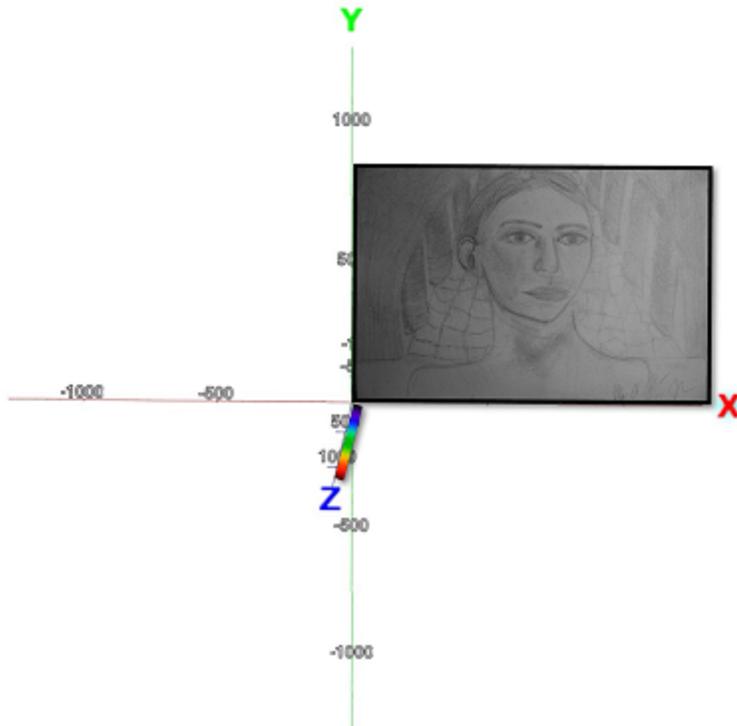
It is important to highlight, that many colors can be obtained from the mixture of primary, secondary and complementary colors. The primary colors are 3, yellow, cyan and magenta. Mixtures of these may vary depending on the mixing ratio, resulting in a wide range of colors.

In conclusion our vector for the visual arts case is a vector (l, w, t) where the variables are length, width and tone. In summary, there are two coordinates which you can locate the point and the wavelength (tone) that ranges from 380nm to 750nm ..

This space would be represented in the following Cartesian axis.



Any technique, either in pencil, oil, tempera can be placed on a plane as above. When using an example like this where it is done on a pencil or pilot of the same color, then there are variations in brightness, but, in any case, they are still wavelengths.



Marco Perez, The witness of reason

This representation in the Cartesian coordinate system provides information in terms of how we analyze the traces but not on their use in creating an artistic form. It is not practical for the artist.

In music we saw that the acoustic form tends to be a manipulation of the notes, which at least in the tonal function, results in the melodic curve. This curve can be approximated by polynomial approximation methods. It is also the case of punctual values, or glissando, which is a continuous exponential function and, because it is not linear, has a certain curvature.

The art of painting is the art of brush stroke, pencil stroke and points. The brush stroke is the path of the pen or brush that goes from point A to a point B. The pencil stroke tends to be thinner than brush strokes, but stroke behavior still tends to be the same. Arguably, the phrase fragment or motif is to music what the stroke is to painting.

For Kerlyn Chavarria's drawing we can extract a segment of hair where you can see the pilot line or path.



In each of the strokes of the drawing, their behavior can be seen as defined logarithms from a point a to point b.

It does not have to be this way, but it can mean that certain behaviors match logarithmic behavior. This is just an example. The hair represented by strokes in the drawing can be estimated. Or simpler functions may be found that do not require interpolation: processes that approximate to the actual behavior. This are segments of a curve. There is a technique where the artist draws the pencil almost horizontally to make a shadow. This technique is sometimes a solid line where it creates a continuous stroke of the pencil. It can be seen as the equivalent of a glissando in music.



This graph closely resembles that behavior and the equations used were:

$$w = \ln(5l - 10) \text{ blue line}$$

$$w = \ln(8l - 20) \text{ yellow line}$$

$$w = \ln(4l - 10) \text{ green line}$$

These are written in terms of  $l$  and  $w$ , the two variables for the location, but equally can be parameterized in terms of  $p$  and use its three variables.

For this case the formula is:

$$P(p) = l(p) + w(p) + t(p)$$

The graphic representation is in 2 dimensions. This are not parametric equations of the form

$f(x) = x$ . This facilitates their comprehensibility and each curve has a different color to differentiate them. This is somewhat similar to what it was done in music in previous chapters.

The equations that are used do not represent nor they are a model of that segment, they only establish a similarity. The conclusion is that the brush or pencil stroke is similar to the fragment pattern in music phraseology, piecewise functions.

In the interpolation chapter it was explained that long segments tend to be very difficult to interpolate. This is best seen in the visual arts, it is impossible to interpolate a drawing as shown on a single algorithm using currently available mathematical tools. They must always be modeled using piecewise functions.

In principle, both arts tend to behave similarly when computers are not used. They are curves.

The music has a tendency to punctual values (extended by the floor function), except in the case of the glissando which is a fragment of a continuous function and where most of the dynamics tend to be static except for some styles such as integral serialism.

In the pencil drawing for example, the strokes are continuous fragments.

What is interesting to observe in the above example, is its simplicity. It is easier to appreciate the development of the curves.

Both in music and painting, these results can be seen as outlines, silhouettes.

Modern art has tools that were not available before, computers, allowing us to extend the possibilities which we did not have before, as regards restrictions on the construction of the instruments. We would be able to make a distinction between the arts.

The natural and synthetic arts.

The synthetic ones are all those which use a computer, since the construction of music by manipulating waves that cannot be performed on the instrument or by increasing the range of the instrument, to the visual arts built from programs that could be designed for this.

Natural arts, although reduced in certain aspects, involve a particular creation, that comes directly from personal efforts. The computer is able to simplify tasks and to do things that humans can not and where you can manipulate more properties of the sound, or visual effect, and it increases accuracy and other important aspects. So this is where the synthetic name is born. In the case of natural arts, the reduced number of variables is very interesting because it implies that at a basic level, humans are capable of handling a maximum of 3 variables. It is easier to manipulate 2 or 3 than to manipulate 4 or 5 which would be a much more complex aspect.

We are able to memorize and copy in a limited way. A painting is particular to its creator and will always have many differences even from their best forgeries. For objects we see a part and

not the whole. At a microscopic level when a stone is painted humans are not capable of representing the millions of visible spectrum frequencies contained in the surface. We can only represent a few, a silhouette. We are able to reproduce art at a macroscopic level. What the eye perceives in a certain distance.

The memory stores a few sequences of events or specific images. Something like photos or sequence of them.

We are not able to memorize each and every one of the events that make up our lives. Only fragments, the most significant for each one. The memorized fragments are the entire personal world, and they are the motor of individual perspective, of actions and reactions.

Language is presented in two ways, oral and written forms, as in music. A person does not need to know how to write in order to articulate words. Just as in music, a person does not need to know how to write in order to play an instrument. Another issue is in the case of deaf-and dumb people who are not required to speak, they can express themselves using signs (sign language – [LESCO in Costa Rica]).

These aspects are related but independent, we can represent its variables.

Spoken language consists of several variables:

- 1) Frequency Changes
- 2) Changes in the intensity (stresses)
- 3) Pronunciation of the sound,
- 4) A location in time

Viewed in representable terms, these variables take the form of:

- a) The allophones: They belong to the branch of phonetics, and refers to the acoustic and physiological aspect of sound production.
- b) Location of the allophone in time. Each sound has issued an order or sequence that gives meaning to the word.
- c) an intensity associated with the intonation of certain words.

While the frequency is essential, this is particular to each person, and varies in a range, but everyone usually produce different frequencies.

### **Written language:**

- a) signs: letters of the alphabet.
- b) a location of the signs in time: These are arranged depending on the word.
- c) intensity: the syllable that is stressed in the word.

Both allophones and letters are able to permutate, a characteristic aspect of languages. This is because language is a limited set of signs defined as allophones. They could be infinite but human capabilities require a restriction on the number of language elements.

On the issue of memory and dreams, we have seen that both music and painting have 3 variables which can be manipulated. Formal languages also have their variables.

*To imagine a melody while talking, involves creating "sounds overlays". It involves imagining a musical segment in two voices, sounding simultaneously. This requires a person trained in music. In reality both the music and spoken language tend to behave in the same medium. They are mechanical waves in the end. The simultaneous perception of the two is not as easy as is required to accommodate each constituent element. It is easier to think of an image or sequence of these and listen to background sounds.*

*During sleep it is much more complex. Memories are involved. These may be similar to reality or mixed together. In dreams variables have greater freedom to mix. There is a much more complex and uniform mixture in each layer. We will call each structure a separate layer. The structures are as follows:*

*musical structure, sounds and spoken language.*

*visual structure*

*written language structure*

*structure of smells.*

*Some say that in dreams we cannot read. Despite such assertion there is brief curiosity. If the above structures are present in dreams, then, to propose a dream analysis is not feasible in terms of modern techniques. This is because, the layers having converging behaviors, it is not known if they exchange issues at some point, or if they take structural patterns and move them towards each other. It seems that music is built on a mould and painting is constructed on a similar mould. What changes is what each variable represents.*

*Then you can manipulate your perceptions within the possibilities offered by the mould (3-structure variables). If you could record dreams, their interpretation is not as simple as some of the proposals suggest. Any possible explanation for this is quite complex.*

*There are many things involved and we have no clear definition as to what they are.*

*All language is representative. Music refers to sounds and properties that can be manipulated. In formal languages, it refers to the object. It is able to represent the object to the imagination. It represents it in a comprehensible way and allows you to create expressions. This is how music and our spoken and written language are quite similar. This is because one transfers characteristics to the other.*

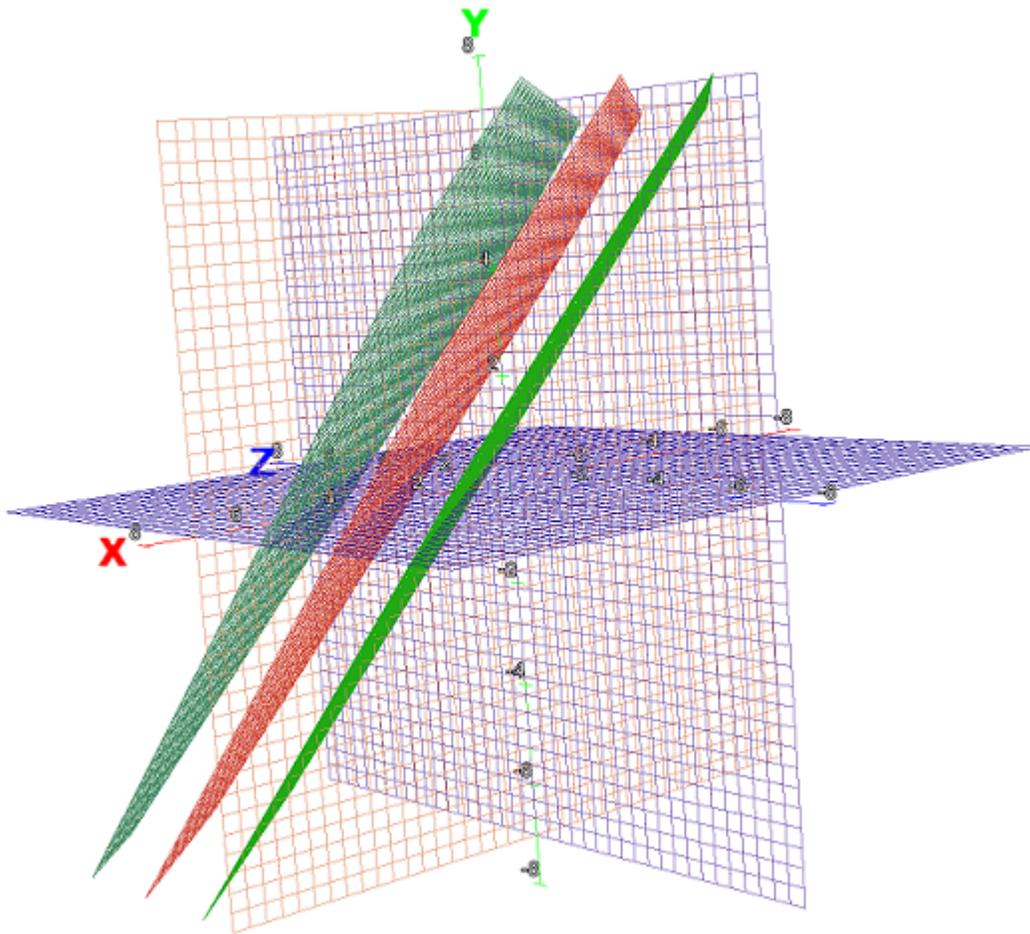
*[note: Marco, the paragraphs that followed in italics appear to repeat paragraphs above so I have deleted them]*

Reality is continuous, a person lives a segment of this, because sooner or later dies and, from the period in which he lived, only a few events are recorded in memory.

- a) each person has experiences and those most relevant to the person are recorded in memory.
- b) The memory is the set of fragments.
- c) The memories are these fragments mentioned. They may be continuous, piecewise defined, or specific, are small pieces of reality, in which we are interested in some way.

*In a waking state people can recall images, sounds, words, or mixtures thereof. For normal people, imagining a sequence of images or musical sounds, plus a parallel dialogue, it is not so simple, and is usually short. Normally it can be imagining the music or imagining the dialogue, but they are difficult to mix simultaneously. There is a kind of interference of ideas for the untrained person, it becomes more difficult if the background music is sung and not just instrumental.*

*In dreams you can have a conversation, listen to music, view pictures, smell odors at the same time or separately. When we dream we are in a state of greater concentration, having the ability to mix structures more freely than in a waking state. So when we dream, all aspects of real life through memory can occur separately or mixed. Their behaviors, as we have seen, are quite similar. Things are not necessarily as experienced in reality. The structures are not necessarily planes, the behavior may vary, but they are a way of representing what is perceived through the senses. This will be something like:*



Each plane is out of phase and represents the mixture of ideas. Even though it is not representative it works to express the idea. In real life, taking a physiological assumption, a human being is able to speak and to perceive smells. But between each of them there is a cut-off point. If we talk we can not smell and when we stop talking during the pause when we catch our breath, then we are able to perceive smells. These sensations are perceived intermittently. Some people refer to dreams as a temporary extra-corporeal state, outside of time. Clearly while we dream we are still subject to time, but what is perceived, does not always seem to be governed by the rules of reality. It is a possible state, based upon logics, since we can not think the impossible.

The overall goal was to give a sense of how music, and visual arts behave. What are its components? Once defined this can initiate a comprehensive study of how the brain handles what it perceives. It is here, at this very point, where a big difference is created, which lies in saying something or saying nothing. The branch of psychology that studies music and color does not say anything. If arguments relating to formal languages are transferred to musical language, it is necessary to ask which of these should be transferred. If there is logic in formal languages, what logic is there in music?

Mathematics is the permitted reasoning, what is possible, and how we understand it.

Physics will provide the explanation for the phenomena based on how we understand them.

In what we are trying to understand we talk about whether or not it has an application and how precise will it be.

There is no accuracy, only approximations.

We are able to see, in a certain way. The visual relates to how we see, if we had a wider viewing angle our view would be different. If the properties of our ears were different, we could hear differently. The senses react to environmental features, aspects that we are able to capture.

Whether or not other features exist, which we do not react to, we cannot ever know, and we can not talk about or analyze those.

**TO BE CONTINUE...**

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